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On the mathematical structure of a large class of physical theories

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Fisica matematica. — *On the mathematical structure of a large class of physical theories* [*]. Nota di ENZO TONTI, presentata [**] dal Socio B. FINZI.

RIASSUNTO. — Si mette in evidenza una struttura matematica comune a differenti teorie fisiche allo scopo di costruire un modello matematico valido per molte di esse.

1.1. INTRODUCTION

Many physical theories show formal similarities due to the existence of a common mathematical structure. This structure is independent of the physical contents of the theory and can be found in classical, relativistic and quantum theories; for discrete and continuous systems. In particular many field theories of different tensorial order exhibit such a structure.

Traditionally the study of similar structures is the subject of the mathematical field theory. Its starting point is the existence of an action principle from which field equations are deduced [1]. Field theory is particularly suitable for fundamental theories as electromagnetism, gravitation and quantum theories [1], [2].

Our approach in this paper is based on a different point of view: we ascertain the existence of a *decomposition* of the fundamental equation into three sets of equations. Typically these are *balance*, *definition* and *constitutive equations*. A deeper insight is so gained which reveals the underlying structure of the fundamental equation: in particular this allows us to ascertain the existence of an action principle for the fundamental equation. In some theories as electromagnetism and continuum mechanics the three sets of equations are more primitive than the fundamental equation obtained combining them. In other theories, as in quantum field theories, the fundamental equation is more primitive and a decomposition can often be achieved.

In the context of analytical mechanics and field theories the canonical form of the fundamental equation is an example of decomposition. We shall show that whenever a canonical decomposition can be achieved it is also possible to obtain the decomposition in the three sets of equations as stated above. But we shall see also that the last decomposition is more general, being possible also when a canonical decomposition cannot be obtained [8].

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(1) In the sequel field equations, wave equations and equations of motion will be denoted with the unique name *fundamental equations*.

1.2. COMPARING DIFFERENT PHYSICAL THEORIES

In every physical theory we can evidentiate some *configuration variables* that describe the configuration of the system or of the field and some *source variables* that describe the sources of the phenomenon. So for ex. lagrangian coordinates in analytical mechanics, the displacement vector in mechanics of continua, the electromagnetic potentials in electromagnetism are examples of configuration variables. Forces, charges, currents, heat production density are examples of source variables.

Besides these two quantities we can find in many theories some intermediate variables as velocities and momenta in mechanical theories, stress and strain tensors in continuum mechanics, the two electromagnetic tensors in electromagnetism, fluxes and affinities in irreversible thermodynamics. A characteristic of these intermediate variables is that they appear always in pairs, those of every pair having the same tensorial order that is not inferior to that of configuration and source variables (that, in turn, have the same tensorial order).

The introduction of intermediate variables is linked to the existence of three kinds of equations which compose the fundamental equation. They are

a) *definition equations*, that define some intermediate variables by means of first order partial or total derivatives of the configuration variables, typically in form of gradients in field theories. We shall denote these variables as *variables of first kind*. In some theory definition equations are nonlinear as in analytical mechanics (see Table I) and in large displacement theory of continuum mechanics (see eq. 1.4.7). Definition equations do not contain material parameters nor physical constants.

b) *balance equations*, which relate source variables to some other intermediate variables that we shall call of *second kind* by means of time derivatives or divergences. They are the local formulation of a global balance from which they are obtained by using Gauss divergence theorem and its generalization for tensors of any order. Balance equations are linear in the variables of second kind even when the associated definition equations are nonlinear (see for ex. those of analytical mechanics in Table I and that of large displacement theory given by eq. (1.4.10)).

c) *constitutive equations*, that relate the variables of first kind to those of second kind and directly source with configuration variables. Typically they contain characteristic parameters of the media and physical constants.

Table I shows the three kinds of equations for some physical theory. Connected with these three kinds of equations there are two more kinds of equations we shall speak of now. In field theories there are variables of first kind that are greater in number than configuration variables and then they are linked by some compatibility conditions. Examples are Saint-Venant compatibility conditions in continuum mechanics, the condition curl $\vec{E}$ in electrostatics,

TABLE I.

Theory	Balance equations	Constitutive equations	Definition equations
Particle dynamics (classical)	$\frac{\mathrm{d}p_h}{\mathrm{d}t} = f_h$	$p_h = m a_{hk} v^k$ (*)	$v^k \overset{\mathrm{def}}{=} \frac{\mathrm{d}x^k}{\mathrm{d}t}$
(relativistic)	$\frac{\mathrm{d}p_h}{\mathrm{d}t} = f_h$	$p_h = \frac{m_0\, a_{hk}}{\sqrt{1 - \frac{v_h v^h}{c^2}}} v^k$	$v^k \overset{\mathrm{def}}{=} \frac{\mathrm{d}x^k}{\mathrm{d}t}$
Space-time formulation	$\frac{\mathrm{d}P_\alpha}{\mathrm{d}\tau} = K_\alpha$	$P_\alpha = m_0\, g_{\alpha\beta}\, U^\beta$	$U^\beta \overset{\mathrm{def}}{=} \frac{\mathrm{d}X^\beta}{\mathrm{d}\tau}$
Analytical mechanics	$-\sum_1^n{}_k \frac{\partial X^i}{\partial q^k} \frac{\mathrm{d}}{\mathrm{d}t} P_i = Q_k$	$P_i = \sum_1^{3N}{}_j m_{ij} V^j$	$V^j \overset{\mathrm{def}}{=} \sum_1^n{}_k \frac{\partial X^j}{\partial q^k} \frac{\mathrm{d}}{\mathrm{d}t} q^k$
Elastodynamics . .	$-\nabla^k p_{hk} + \frac{\partial p_h}{\partial t} = f_h$	$\begin{cases} p_{hk} = C_{hkrs}\, e^{rs} \\ p_h = \rho a_{hk} v^k \end{cases}$	$\begin{cases} e^{rs} \overset{\mathrm{def}}{=} \frac{1}{2}(u^{r/s} + u^{s/r}) \\ v^k \overset{\mathrm{def}}{=} \frac{\partial u^k}{\partial t} \end{cases}$
Viscoelasticity . . .	$-\nabla^k p_{hk} = f_h$	$p_{hk} = \int_0^t C_{hkrs}(t-\tau) \cdot \frac{\partial}{\partial\tau} e^{rs}(\tau)\, \mathrm{d}\tau$	$e^{rs} \overset{\mathrm{def}}{=} \frac{1}{2}(u^{r/s} + u^{s/r})$
Electromagnetism .	$\begin{cases} \nabla^h D_h = \rho \\ -\frac{\partial D_h}{\partial t} + \varepsilon_{hr}{}^s \nabla^r H_s = J_h \end{cases}$	$\begin{cases} D_h = \varepsilon_{hk} E^k \\ H_s = \nu_{sl} B^l \end{cases}$	$\begin{cases} E^k \overset{\mathrm{def}}{=} \frac{\partial A^k}{\partial t} - \nabla^k \varphi \\ B^l \overset{\mathrm{def}}{=} \varepsilon^{li}{}_j \nabla_i A^j \end{cases}$
(Space-time formulation)	$\nabla^\alpha f_{\alpha\beta} = J_\beta$	$f_{\alpha\beta} = G_{\alpha\beta\rho\sigma} F^{\rho\sigma}$	$F^{\rho\sigma} \overset{\mathrm{def}}{=} \Phi^{\rho/\sigma} - \Phi^{\sigma/\rho}$
Irreversible thermodynamics	$\mathbf{div}\, \mathbf{J}_k = \sigma_k$	$\mathbf{J}_k = \sum_j L_{kj} \mathbf{F}_j$	$\mathbf{F}_j \overset{\mathrm{def}}{=} \mathbf{grad}\, Z_j$
Heat conduction. .	$\nabla^h q_h + \sigma_t = \sigma$	$\begin{cases} q_h = \lambda(T)\, a_{hk} p^k \\ \sigma_t = \rho c_v \frac{\partial}{\partial t} T \end{cases}$	$p^k \overset{\mathrm{def}}{=} -\nabla^k T$
Quantum mechanics (Schrödinger) . .	$\nabla^h q_h + \sigma_t = \sigma$	$\begin{cases} q_h = \frac{\hbar^2}{2m} a_{hk} v^k \\ \sigma_t = -i\hbar \frac{\partial}{\partial t} \psi \end{cases}$	$v^k \overset{\mathrm{def}}{=} -\nabla^k \psi$
(Klein-Gordon) . .	$\nabla^\alpha q_\alpha + \tau = \sigma$	$\begin{cases} q_\alpha = \hbar^2 g_{\alpha\beta} v^\beta \\ \tau = (m_0 c)^2 \psi \end{cases}$	$v^\beta \overset{\mathrm{def}}{=} \nabla^\beta \psi$

(*) a_{hk} denotes the metric tensor in the three dimensional space; $g_{\alpha\beta}$ that of space-time. Latin indexes range between 1 and 3, Greek indexes range between 0 and 3. Tensor indexes are summed when repeated.

one set of Maxwell equations (see eq. 1.2.5) in electromagnetism. If φ denotes the set of configuration variables, u that of variables of first kind, $\mathfrak{D}$ denotes the formal differential operator of definition equation, the last can be written

$$u = \mathfrak{D}\varphi \,. \tag{1.2.1}$$

In many cases the vanishing of φ does not imply the vanishing of u because there are impressed u_0, i.e.

$$u = \mathfrak{D}\varphi + u_0 \,. \tag{1.2.2}$$

In this case the compatibility conditions can be written in the form

$$\mathfrak{R}u = \tau \qquad (\tau \stackrel{\text{def}}{=} \mathfrak{R}u_0 \; ; \; \mathfrak{R}\mathfrak{D} = 0) \tag{1.2.3}$$

and the term τ appears as a source of "incompatibility". Typical is the incompatibility tensor η_{li} in continuum mechanics given by

$$\varepsilon_{lhr}\, \varepsilon_{iks}\, \nabla^h \nabla^k e^{rs} = \eta_{li} \tag{1.2.4}$$

used in the theory of dislocations [3]. Another example is that of the postulated magnetic current density (the Dirac monopole) in Maxwell equations

$$\varepsilon^{\alpha\beta\gamma\rho}\, \nabla_\beta\, F_{\gamma\rho} = J^\alpha_{\text{magn.}} \tag{1.2.5}$$

In this case the incompatibility tensor τ of eq. (1.2.3) will be called *dual source variable* and eq. (1.2.3) *dual balance equation*. In theories where the configuration variables are measurable quantities, definition equations are more primitive than dual balance equations: this is the case of heat conduction and continuum mechanics (see Table I). In theories where configuration variables are nonmeasurable quantities the dual balance equations are more primitive than definition equations, as in the case of electromagnetism.

Balance equations in field theories admit a general solution: if σ denotes the source variables, v the variables of second kind and $\mathfrak{B}$ the formal differential operator of balance equations these can be written in the form

$$\mathfrak{B}v = \sigma \,. \tag{1.2.6}$$

The general solution will be of the form

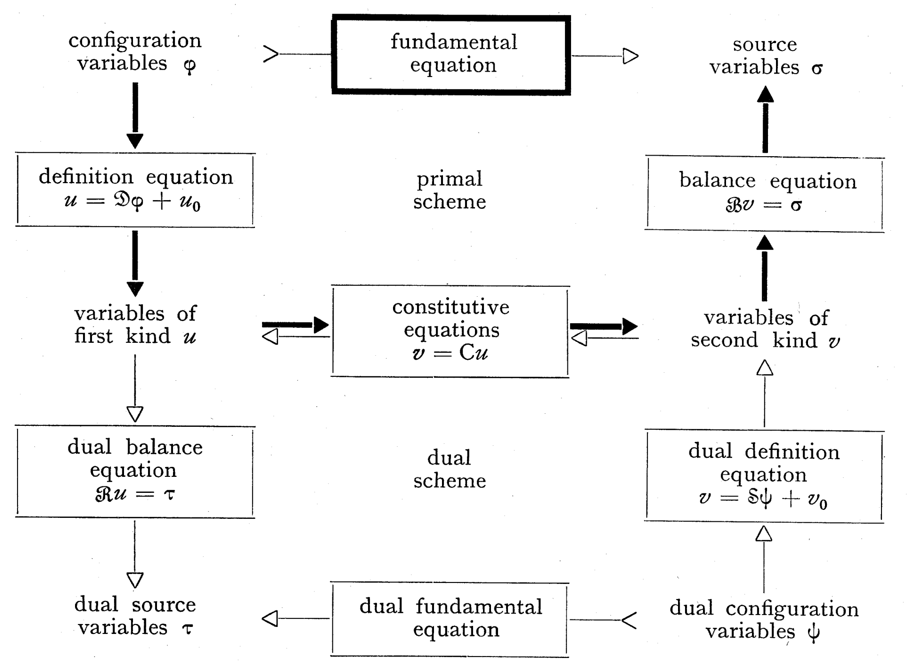
$$v = \mathfrak{S}\psi + v_0 \,. \tag{1.2.7}$$

Because of the resemblance of this equation with definition equation (1.2.2) we shall call eq. (1.2.7) *dual definition equation* and ψ *dual configuration variable*. Examples of such equations are the general solution of the equilibrium equations of statics of continua [4]

$$p_{hk} = \varepsilon_{hil}\, \varepsilon_{kjm}\, \nabla^i \nabla^j \chi^{lm} + p^{(0)}_{hk} \tag{1.2.8}$$

being χ^{lm} the stress potential. The dual definition equation gives the general solution of balance equation that, in turn, expresses the compatibility conditions for the defined variables. The equation that link the dual configuration variables and the dual source variables will be called *dual fundamental equation*. Table II display the relative position of the various variables and equations.

TABLE II.



Up to now we have considered only formal differential operators. A less formal study of the various equations require the consideration of the initial and boundary conditions associated to the equations. In this way we are led to consider operators instead of formal operators.

1.3. Bilinear functionals: their importance

In many physical theories we use bilinear forms formed by products of two variables of the same tensorial order. From these bilinear forms one can obtain bilinear functionals integrating on space and time variables. Examples are shown in Table III. Bilinear functionals play a fundamental part in defining the adjoint of a linear operator. If Φ and Σ are two spaces put in duality by a bilinear functional $\langle \sigma , \varphi \rangle_{\mathrm{I}}$, [5], and U, V are two more spaces put in duality by a bilinear functional $\langle v , u \rangle_{\mathrm{II}}$ then given a linear operator $\mathrm{L} : \Phi \mapsto \mathrm{U}$ its adjoint is defined as the linear operator $\widetilde{\mathrm{L}} : \mathrm{V} \mapsto \Sigma$ such that

$$\langle v , \mathrm{L}\varphi \rangle_{\mathrm{II}} = \langle \widetilde{\mathrm{L}} v , \varphi \rangle_{\mathrm{I}} . \tag{1.3.1}$$

TABLE III.

Bilinear functionals used in some physical theories.

Electrostatics	$\langle \rho , \varphi \rangle = \iiint_V \rho\varphi \, dV$
Electromagnetism	$\langle J , \Phi \rangle = \iiiint_\Omega J_\alpha \Phi^\alpha \, d\Omega$
Statics of continua	$\langle f , u \rangle = \iiint_V f_k u^k \, dV$
Particle mechanics	$\langle F , r \rangle = \int_0^T F_k x^k \, dt$
Classical gravitation	$\langle \rho_m , \varphi \rangle = \iiint_V \rho_m \varphi \, dV$
Relativistic theory of gravitation	$\langle T , g \rangle = \iiiint_\Omega T_{\alpha\beta} g^{\alpha\beta} \, d\Omega$
Thermal field	$\langle \sigma , T \rangle = \iiint_V \sigma T \, dV$
Fluidynamics	$\langle f , v \rangle = \iiint_V \int_0^T f_k v^k \, dV \, dt$
Analytical statics	$\langle Q , q \rangle = \sum_1^n {}_k Q_k q^k$
Analytical dynamics	$\langle Q , q \rangle = \int_0^T \sum_1^n {}_k Q_k q^k \, dt$
Quantum mechanics	$\langle \sigma , \psi \rangle = \iiint_V \int_0^T (\bar{\sigma}\psi + \sigma\bar{\psi}) \, dV \, dt$
Thermostatics	$\langle Y, X \rangle = \sum_1^n {}_k Y_k X^k$

Fig. 1 shows the mutual relation between L and $\widetilde{\mathrm{L}}$.

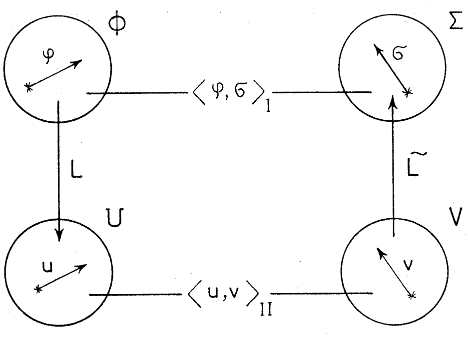


Fig. 1.

We emphasize the fact that to define the adjoint of an operator two bilinear functionals are needed, i.e. two pairs of spaces. The reason why usually only one bilinear functional is used is because we usually work in Hilbert spaces where the bilinear functional reduces to the scalar product of two elements of the same space [6].

1.4. Relation between the operators of balance and definition equations

If in every physical theory a comparison is made between the differential operators that form the equations of definition and the equations of balance one discovers that they are linked by a simple and important relation. We show here some examples.

1) The equations of electrostatics

$$(1.4.1) \qquad \begin{cases} \nabla\cdot\vec{\mathrm{D}} = \rho \\ \vec{n}\cdot\vec{\mathrm{D}} = 0 \quad \text{on } \mathrm{S}_1 \end{cases} \qquad \begin{cases} \vec{\mathrm{E}} = -\nabla\varphi \\ \varphi = 0 \quad \text{on } \mathrm{S}_2 \end{cases}$$

being $\mathrm{S}_1 \cup \mathrm{S}_2 = \mathrm{S}$, are formed with the two operators that are one adjoint of the other with respect to the bilinear functionals

$$(1.4.2) \qquad \langle\rho\,,\varphi\rangle = \iiint_{\mathrm{V}} \rho\,\varphi\,d\mathrm{V} \qquad \langle\mathrm{D}\,,\mathrm{E}\rangle = \iiint_{\mathrm{V}} \mathrm{D}_h\,\mathrm{E}^h\,d\mathrm{V}\,.$$

2) Analogous property is shown by the two equations of statics of continua

$$(1.4.3) \qquad \begin{cases} -\nabla^k p_{hk} = f_h \\ n^k p_{hk} = 0 \quad \text{on } \mathrm{S}_1 \end{cases} \qquad \begin{cases} e^{hk} = \frac{1}{2}\,(u^{h|k} + u^{k|h}) \\ u^h = 0 \quad \text{on } \mathrm{S}_2 \end{cases}$$

whose operators are adjoint one of the other with respect to the two bilinear functionals

$$(1.4.4) \qquad \langle f , u\rangle = \iiint\limits_V f_h\, u^h\, \mathrm{dV} \qquad \langle p , e\rangle = \iiint\limits_V p_{hk}\, e^{hk}\, \mathrm{dV} .$$

In evolution theories where time derivatives enter into balance equations the relation of adjointness between balance and definition operators can be discovered using a convolution bilinear functional [6].

3) For example particle dynamics governed by Newton equation of motion, can be decomposed into the three equations

$$(1.4.5) \qquad \begin{cases} \dfrac{\mathrm{d}p_h}{\mathrm{d}t} = f_h \\ p_h(0) = 0 \end{cases} \qquad p_h = m a_{hk}\, v^k \qquad \begin{cases} v^k = \dfrac{\mathrm{d}}{\mathrm{d}t}\, x^k \\ x^k(0) = 0 \end{cases}$$

that represent respectively balance, constitutive and definition equations.

If we consider the bilinear functionals

$$(1.4.6) \qquad \langle f , x\rangle_c = \int_0^T f_k(t)\, x^k(\mathrm{T}-t)\, \mathrm{d}t \qquad \langle p , v\rangle_c = \int_0^T p_k(t)\, v^k(\mathrm{T}-t)\, \mathrm{d}t$$

in which the convolution of two functions is used, then the operators of the two sets are adjoint one of the other.

4) In some theory in which definition equations are nonlinear the following fact arises: the operator of balance equation is the adjoint of the *derivative* of the operator of definition equation.

So, for ex. in the large displacement theory of continuum mechanics, the Green strain tensor given by (see [7])

$$(1.4.7) \qquad \begin{cases} e^{hk} = \dfrac{1}{2}\,(u^{h/k} + u^{k/h} + u^{l/h}\, u_l^{\ /k}) \\ u^h = 0 \quad \text{on } \mathrm{S}_1 , \end{cases}$$

has first variation as

$$(1.4.8) \qquad \delta e^{hk} = \frac{1}{2}\,(\nabla^h\, \delta u^k + \nabla^k\, \delta u^h + \nabla^h\, \delta u^l\, u_l^{\ /k} + u^{l/h}\, \nabla^k\, \delta u_l)$$

that is *linear* in the variations δu^p. Using the two bilinear functionals (1.4.4) we obtain the identity

$$(1.4.9) \qquad \iiint\limits_V p_{hk} \left[\frac{1}{2}\,(\nabla^h \delta u^k + \nabla^k \delta u^h + \nabla^h \delta u^l\, u_l^{\ /k} + u^{l/h} \nabla^k \delta u_l)\right] \mathrm{dV} \equiv$$

$$\equiv \iiint\limits_V -\nabla^k\,[(\delta_l^h + u_l^{\ /h})\, p_{hk}]\, \delta u^l\, \mathrm{dV} + \oint\!\!\oint\limits_S n^k\,(\delta_l^h + u_l^{\ /h})\, p_{hk}\, \delta u^l\, \mathrm{dS} .$$

When boundary conditions on u^l are taken into account one obtains as adjoint operator that of equilibrium equation

$$\left\{ \begin{array}{l} -\nabla^k (\delta_l^h + u_l^{\ |h})\, p_{hk} = f_l \\ n^k (\delta_l^h + u_l^{\ |h})\, p_{hk} = 0 \quad \text{on } S_2 \end{array} \right. \tag{1.4.10}$$

being p_{hk} the Kirchhoff stress tensor [7].

Summarizing we can say that *in many physical theories the operator of balance equation and that of definition equation are mutually adjoint with respect to some bilinear form (when the last is linear); while if the definition equation is nonlinear the operator of balance equation is adjoint of the derivative of the definition operator*. This relation of adjointness plays a fundamental part in the structure of the mathematical model of physical theories we shall build up in other papers [8].

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