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Curvature invariants associated with fundamental forms of a Fins1er hypersurface

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Geometria differenziale. — *Curvature invariants associated with fundamental forms of a Finsler hypersurface.* Nota di U. P. SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — H Rund ha definito [1] n forme fondamentali per una ipersuperficie in uno spazio di Riemann n–dimensionale.

Analoga ricerca è qui fatta per le ipersuperficie di uno spazio di Finsler n–dimensionale nel caso siano soddisfatte ulteriori condizioni semplificatrici.

1. FUNDAMENTAL FORMULAE

Let a hypersurface F_{n-1} given by the equations

$$x^i = x^i(u^\alpha); \qquad i = 1, \cdots, n \;;\; \alpha = 1, \cdots, n-1$$

be immersed in a n–dimensional Finsler space F_n. The components $\dot{x}^i$ and $\dot{u}^\alpha$ of a vector [of magnitude $F(x, \dot{x})$] tangent to F_{n-1} are related by $\dot{x}^i = B^i_\alpha \dot{u}^\alpha$, where $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$. The unit vector l^i along $\dot{x}^i$ (or $\dot{x}^i$ itself) is taken to be the element of support. The metric tensors $g_{ij}(x, \dot{x})$ and $g_{\alpha\beta}(u, \dot{u})$ of F_n and F_{n-1} are related by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})\, B^i_\alpha B^j_\beta. \tag{1.1}$$

The normal vector $N^i(x, \dot{x})$ of the hypersurface satisfies the conditions

$$g_{ij}(x, \dot{x})\, N^j B^i_\alpha = N_i B^i_\alpha = 0 \quad \text{and} \quad g_{ij}(x, \dot{x})\, N^i N^j = 1. \tag{1.2}$$

For a displacement vector dx^i and a vector-field X^i, both tangent to F_{n-1}, we may write

$$dx^i = B^i_\alpha\, du^\alpha \quad , \quad X^i = B^i_\alpha X^\alpha. \tag{1.3}$$

If DX^i stands for the Cartan's covariant differential, then the 'induced' differential is defined by

$$\bar{D}X^\alpha = B^\alpha_i\, DX^i \tag{1.4}$$

where

$$B^\alpha_i = g^{\alpha\varepsilon}(u, \dot{u})\, g_{ij}(x, \dot{x})\, B^j_\varepsilon.$$

The connection parameters Γ^{*i}_{hk}, $\Gamma^{*\varepsilon}_{\alpha\beta}$ and tensor $I^i_{\alpha\beta}$ (defined in [2] pages 68, 160 and 193 respectively) may be used in defining

$$\Omega_{\alpha\beta} = N_i\, I^i_{\alpha\beta}. \tag{1.5}$$

The tensor $\Omega_{\alpha\beta}$ is called the second fundamental tensor of the hypersurface.

(*) Nella seduta del 15 gennaio 1972.

We quote the following results (Rund [2] pages 158–165) for reference in the later sections of this paper.

Defining

$$\text{(1.6)} \qquad N^i_j = N^i N_j \quad , \quad A^i_{kh} = FC^i_{kh} \quad , \quad C_{\alpha\beta\gamma} = g_{\beta\varepsilon} C^\varepsilon_{\alpha\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma$$

and

$$\text{(1.7)} \qquad \bar{\Gamma}^{*\alpha}_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - C^\alpha_{\beta\delta} \Gamma^\delta_{\varepsilon\gamma} \dot{u}^\varepsilon$$

it has been shown

$$\text{(1.8)} \qquad Dl^i - B^i_\alpha \bar{D} l^\alpha = H^i_\gamma du^\gamma$$

and

$$\text{(1.9)} \qquad DX^i - B^i_\alpha \bar{D} X^\alpha = H^i_{\gamma\beta} X^\beta du^\gamma + N^i_j A^j_{hk} B^h_\beta B^k_\gamma X^\beta \bar{D} l^\gamma$$

where we have taken

$$\text{(1.10)} \qquad FH^i_\gamma = (B^i_{\beta\gamma} - B^i_\varepsilon \Gamma^\varepsilon_{\beta\gamma}) \dot{u}^\beta + B^k_\gamma \Gamma^i_{hk} \dot{x}^h$$

and

$$\text{(1.11)} \qquad H^i_{\gamma\beta} = (B^i_{\beta\gamma} - B^i_\alpha \bar{\Gamma}^{*\alpha}_{\beta\gamma} + \Gamma^{*i}_{hk} B^h_\beta B^k_\gamma + A^i_{hk} B^h_\beta H^k_\gamma)$$

(the symbols C_{ijk} , $\Gamma^\alpha_{\beta\gamma}$ have their usual meanings [2]).

Rund [3] has introduced certain tensors which vanish identically in any locally Euclidean (or Riemannian) theory of hypersurface. These tensors and associated quantities are defined by

$$\text{(1.12)} \qquad M_{ij}(x , \dot{x}) = C_{ijk}(x , \dot{x}) N^k \quad , \quad M_i(x , \dot{x}) = C_{ijk} N^j N^k$$

and

$$\text{(1.13)} \qquad M_{\alpha\beta} = C_{ijk}(x , \dot{x}) B^i_\alpha B^j_\beta N^k \quad , \quad M_\alpha = C_{ijk}(x , \dot{x}) B^i_\alpha N^j N^k .$$

The following relations are obvious.

$$\text{(1.14)} \qquad M_{ij}(x , \dot{x}) \dot{x}^j = 0 \quad , \quad M_i(x , \dot{x}) \dot{x}^i = 0 ,$$

$$M_{\alpha\beta}(u , \dot{u}) \dot{u}^\beta = 0 \quad \text{and} \quad M_\alpha(u , \dot{u}) \dot{u}^\alpha = 0 .$$

2. n Fundamental forms of F_{n-1}

Let $C : u^\alpha = u^\alpha(\lambda)$ be a curve (not in the direction of $\dot{u}^\alpha$) of the hypersurface. The components in x^i and u^α of the tangent vector of this curve are such that

$$\text{(2.1)} \qquad \frac{dx^i}{d\lambda} = B^i_\alpha \frac{du^\alpha}{d\lambda} .$$

This vector is normalised by the condition

$$(2.2) \qquad g_{ij}(x, \dot{x}) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = g_{\alpha\beta}(u, \dot{u}) \frac{du^\alpha}{d\lambda} \frac{du^\beta}{d\lambda} = 1 .$$

Considering the displacement du^γ in (1.9) along C and putting $X^\alpha = \frac{du^\alpha}{d\lambda}$ we obtain

$$(2.3) \qquad \frac{D\xi^i}{D\lambda} - B^i_\alpha \frac{\bar{D}\xi^\alpha}{D\lambda} = H^i_{\gamma\beta}(u, \dot{u}) \xi^\beta \xi^\gamma + FN^i M_{\beta\gamma}(u, \dot{u}) \xi^\beta \frac{\bar{D}l^\gamma}{D\lambda},$$

where for brevity we have taken $\xi^i = \frac{dx^i}{d\lambda}$, $\xi^\alpha = \frac{du^\alpha}{d\lambda}$. It may be verified with the help of equation (1.4) that $\left(\frac{D\xi^i}{D\lambda} - B^i_\alpha \frac{\bar{D}\xi^\alpha}{D\lambda}\right)$ is along the normal to the hypersurface and this may be used in obtaining

$$(2.4) \qquad K\left(u, \dot{u}, \frac{du}{d\lambda}\right) = N_i\left(\frac{D\xi^i}{D\lambda} - B^i_\alpha \frac{\bar{D}\xi^\alpha}{D\lambda}\right) =$$

$$= H_{\gamma\beta}(u, \dot{u}) \xi^\beta \xi^\gamma + FM_{\beta\gamma}(u, \dot{u}) \xi^\beta \frac{\bar{D}l^\gamma}{D\lambda},$$

where in view of (1.11), (1.10) and relations $\Gamma^\varepsilon_{\beta\gamma} \dot{u}^\beta = \Gamma^{*\varepsilon}_{\beta\gamma} \dot{u}^\beta$; $\Gamma^i_{hk} \dot{x}^h = \Gamma^{*i}_{hk} \dot{x}^h$,

$$(2.5) \qquad H_{\beta\gamma} = N_i H^i_{\beta\gamma} = \Omega_{\beta\gamma} + M_\gamma \Omega_{\beta\delta} \dot{u}^\delta .$$

An expression for induced mixed covariant derivative $N^i_{\|\gamma}$ has been obtained by Rund [3]. This expression when substituted in $DN^i = N^i_{\|\gamma} du^\gamma$ will reduce it to

$$(2.6) \qquad DN^i = -\Omega_{\sigma\gamma} g^{\sigma\varepsilon} B^i_\varepsilon du^\gamma + (N^i M_l - 2 M^i_l) I^l_{\sigma\gamma} \dot{u}^\sigma du^\gamma .$$

In view of this relation and equations (1.5), (1.13) we obtain

$$(2.7) \qquad -g_{ij} DN^i dx^j = [\Omega_{\gamma\delta} + 2 M_\delta \Omega_{\sigma\gamma} \dot{u}^\sigma] du^\gamma du^\delta .$$

Using the equation (2.6) once again and simplifying with the help of (1.2), (1.12) and (1.13) we obtain

$$(2.8) \qquad g_{ij} DN^i DN^j = [\Omega_{\theta\gamma} \Omega_{\rho\delta} g^{\rho\theta} + 4 M_\varepsilon \Omega_{\sigma\gamma} g^{\sigma\varepsilon} \Omega_{\rho\delta} \dot{u}^\rho +$$

$$+ (4 M_i M^i - 3 M^2) \Omega_{\sigma\gamma} \Omega_{\rho\delta} \dot{u}^\sigma \dot{u}^\rho] du^\gamma du^\delta$$

where

$$M = M_i N^i .$$

The expressions (2.7) and (2.8) are too complicated to make further study possible. We, therefore, assume that at x^i the element of support $\dot{x}^i$ satisfies the condition

$$(2.9) \qquad M_{ij}(x, \dot{x}) = C_{ijk}(x, \dot{x}) N^k(x, \dot{x}) = 0 .$$

This implies $M_i = 0$, $M_{\alpha\beta} = 0$, $M_\alpha = 0$. The equations (2.7) and (2.8) will now reduce to

$$-g_{ij}(x, \dot{x})\, DN^i\, dx^j = C_{(2)\alpha\beta}(u, \dot{u})\, du^\alpha\, du^\beta \tag{2.10}$$

and

$$g_{ij}(x, \dot{x})\, DN^i\, DN^j = C_{(3)\alpha\beta}(u, \dot{u})\, du^\alpha\, du^\beta \tag{2.11}$$

where

$$C_{(2)\alpha\beta} = \Omega_{\alpha\beta} \quad \text{and} \quad C_{(3)\alpha\beta} = C_{(2)\alpha\varepsilon}\, \Omega^\varepsilon_\beta . \tag{2.12}$$

Using the above equation and following the method of Rund [1] we write down the recurrence formula

$$C_{(p)\alpha\beta}(u, \dot{u}) = C_{(p-1)\alpha\varepsilon}(u, \dot{u})\, \Omega^\varepsilon_\beta(u, \dot{u}) \tag{2.13}$$

$C_{(p)\alpha\beta}(u, \dot{u})$ will be called the coefficients of the *p–th* fundamental form of F_{n-1}.

3. Principal directions

The conditions $M_{ij}(x, \dot{x}) = 0$ and equations (2.4), (2.5) yield

$$K(u, \dot{u}, \xi) = \Omega_{\beta\gamma}(u, \dot{u})\, \xi^\beta\, \xi^\gamma. \tag{3.1}$$

Since $K(u, \dot{u}, \xi)$ is a quadratic expression in ξ, the equations determining principal directions will be linear. They will therefore yield $(n-1)$ principal directions of the hypersurface. It is now possible to extend the theory developed in [1] to the hypersurface of a Finsler space.

In particular, if $\lambda_1, \lambda_2 \cdots \lambda_{n-1}$ are principal curvatures of the hypersurface then we define *p–th* curvature H_p of F_{n-1} in the form

$$H_p = \Sigma \lambda_1 \lambda_2 \cdots \lambda_p = \text{Sum of products (taken } p \text{ at a time) of the principal curvatures} \quad (p = 1, \cdots, n-1). \tag{3.2}$$

The *p–th* associated mean curvature of F_{n-1} is defined as

$$M_p = g^{\alpha\beta}\, C_{(p+1)\alpha\beta}. \tag{3.3}$$

Using the method of [1] we deduce

$$H_1 = g^{\alpha\beta}\, \Omega_{\alpha\beta} = M_1, \tag{3.4}$$

$$H_2 = \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} (\Omega^r_r\, \Omega^{r+s}_{r+s} - \Omega^r_{r+s}\, \Omega^{r+s}_r) \tag{3.5}$$

(no summation with respect to r and s) and

$$2\, H_2 = M_1^2 - M_2. \tag{3.6}$$

4. INDUCED AND INTRINSIC CURVATURE TENSORS

In view of the results obtained above, it is possible to express the curvature invariant H_2 defined by (3.2) in terms of the curvature tensors of F_n and F_{n-1}. In the existing literature two types of curvature tensors have been defined for a Finsler hypersurface (Rund [3]). These are called induced and intrinsic curvature tensors. The generalised equations of Gauss have been obtained for both the curvature tensors. The generalised Gauss equation for the induced curvature tensor $K_{\alpha\beta\gamma\varepsilon}$ is given by (Rund [4])

$$(4.1)\qquad \begin{aligned} K_{\alpha\beta\gamma\varepsilon} = {} & K_{ijhk}\, B^i_\alpha\, B^j_\beta\, B^h_\gamma\, B^k_\varepsilon + (\Omega_{\alpha\gamma}\,\Omega_{\beta\varepsilon} - \Omega_{\alpha\varepsilon}\,\Omega_{\beta\gamma}) + \\ & + 2\, C_{jhl}\, B^j_\beta\, N^h\, (I^l_{\sigma\varepsilon}\,\Omega_{\alpha\gamma} - I^l_{\sigma\gamma}\,\Omega_{\alpha\varepsilon})\, \dot{u}^\sigma + \\ & + g_{ij}\, B^j_\beta\, B^h_\alpha\, (\partial\Gamma^{*i}_{hk}/\partial\dot{x}^l)\, (I^l_{\sigma\varepsilon}\, B^k_\gamma - I^l_{\sigma\gamma}\, B^k_\varepsilon)\, \dot{u}^\sigma, \end{aligned}$$

where K_{ijhk} is the curvature tensor of F_n.

The intrinsic curvature tensor $\hat{K}_{\alpha\beta\gamma\varepsilon}$ has been obtained with the help of intrinsic connection parameter $\hat{\Gamma}^\alpha_{\beta\gamma}$ which is defined with respect to the metric of F_{n-1} in a manner formally identical with the mode of definition of Γ^{*i}_{hk} in terms of the metric of F_n. Writing

$$2\, G^\varepsilon = \gamma^\varepsilon_{\alpha\beta}\, \dot{u}^\alpha\, \dot{u}^\beta \quad , \quad G^\varepsilon_\nu = \frac{\partial G^\varepsilon}{\partial \dot{u}^\nu},$$

we have the definition (Rund [3])

$$\hat{K}^\delta_{\alpha\gamma\varepsilon} = \frac{\partial\hat{\Gamma}^\delta_{\alpha\gamma}}{\partial u^\varepsilon} - \frac{\partial\hat{\Gamma}^\delta_{\alpha\varepsilon}}{\partial u^\gamma} - \left(\frac{\partial\hat{\Gamma}^\delta_{\alpha\gamma}}{\partial\dot{u}^\lambda}\, G^\lambda_\varepsilon - \frac{\partial\hat{\Gamma}^\delta_{\alpha\varepsilon}}{\partial\dot{u}^\lambda}\, G^\lambda_\gamma\right) + \hat{\Gamma}^\delta_{\lambda\varepsilon}\,\hat{\Gamma}^\lambda_{\alpha\gamma} - \hat{\Gamma}^\delta_{\lambda\gamma}\,\hat{\Gamma}^\lambda_{\alpha\varepsilon},$$

($\gamma^\varepsilon_{\alpha\beta}$ being the Christoffel's symbols of the second kind for F_{n-1}). The Gauss equation for intrinsic curvature tensor is now given by ([3])

$$(4.2)\qquad \begin{aligned} \hat{K}_{\alpha\beta\gamma\varepsilon} = {} & K_{ijhk}\, B^i_\alpha\, B^j_\beta\, B^h_\gamma\, B^k_\varepsilon + (\Omega_{\alpha\gamma}\,\Omega_{\beta\varepsilon} - \Omega_{\alpha\varepsilon}\,\Omega_{\beta\gamma}) + \\ & + 2\, M_{jl}\, B^j_\beta\, (\Omega_{\alpha\gamma}\, J^l_{\sigma\varepsilon} - \Omega_{\alpha\varepsilon}\, J^l_{\sigma\gamma})\, \dot{u}^\sigma + (\Delta_{\sigma\beta\gamma}\,\Delta^\sigma_{\alpha\varepsilon} - \Delta_{\sigma\beta\varepsilon}\,\Delta^\sigma_{\alpha\gamma}) + \\ & + (\Delta_{\alpha\beta\gamma/\varepsilon} - \Delta_{\alpha\beta\varepsilon/\gamma}) + g_{ij}\, B^j_\beta\, \frac{\partial\Gamma^{*i}_{hk}}{\partial\dot{x}^l}\, B^h_\alpha\, (J^l_{\sigma\varepsilon}\, B^k_\gamma - J^l_{\sigma\gamma}\, B^k_\varepsilon)\, \dot{u}^\sigma, \end{aligned}$$

where

$$(4.3)\qquad \Delta^\varepsilon_{\alpha\beta} = \hat{\Gamma}^\varepsilon_{\alpha\beta} - \Gamma^{*\varepsilon}_{\alpha\beta} \quad , \quad \Delta_{\alpha\gamma\beta} = g_{\gamma\varepsilon}\,\Delta^\varepsilon_{\alpha\beta},$$

$$(4.4)\qquad J^i_{\alpha\beta} = B^i_{\alpha\beta} - B^i_\varepsilon\,\hat{\Gamma}^\varepsilon_{\alpha\beta} + \Gamma^{*i}_{hk}\, B^h_\alpha\, B^k_\beta = I^i_{\alpha\beta} - B^i_\varepsilon\,\Delta^\varepsilon_{\alpha\beta},$$

and $\Delta_{\alpha\gamma\beta/\varepsilon}$ is the intrinsic covariant derivative of $\Delta_{\alpha\beta\gamma}$. In view of the fact ([3])

$$(4.5)\qquad \begin{aligned} \Delta_{\alpha\gamma\beta} = {} & C_{hkj}\, N^j\, [(B^h_\beta\, B^k_\gamma\,\Omega_{\alpha\sigma} + B^h_\alpha\, B^k_\gamma\,\Omega_{\beta\sigma} - B^h_\beta\, B^k_\alpha\,\Omega_{\gamma\sigma})\, \dot{u}^\sigma - \\ & - (C^\lambda_{\beta\gamma}\, B^h_\lambda\, B^k_\alpha + C^\lambda_{\alpha\gamma}\, B^h_\lambda\, B^k_\beta - C^\lambda_{\beta\alpha}\, B^h_\lambda\, B^k_\gamma)]\, \Omega_{\sigma\lambda}\, \dot{u}^\sigma\, \dot{u}^\lambda \end{aligned}$$

and the condition (2.9) we deduce

$$\Delta_{\alpha\gamma\beta} = 0 \quad , \quad \hat{\Gamma}^{\varepsilon}_{\alpha\beta} = \Gamma^{*\varepsilon}_{\alpha\beta} \quad \text{and} \quad J^{i}_{\alpha\beta} = I^{i}_{\alpha\beta}\,.$$

A comparison of equations (4.1) and (4.2) now reveals that if the element of support satisfies the condition (2.9) then each induced and intrinsic generalised Gauss equation will reduce to

$$\text{(4.6)} \qquad K_{\alpha\beta\gamma\varepsilon} = \hat{K}_{\alpha\beta\gamma\varepsilon} = K_{ijhk}\, B^{i}_{\alpha}\, B^{j}_{\beta}\, B^{h}_{\gamma}\, B^{k}_{\varepsilon} + (\Omega_{\alpha\gamma}\,\Omega_{\beta\varepsilon} - \Omega_{\alpha\varepsilon}\,\Omega_{\beta\gamma}) + \\ + g_{ij}\, B^{j}_{\beta}\, B^{h}_{\alpha}\, \frac{\partial \Gamma^{*i}_{hk}}{\partial \dot{x}^{l}}\, N^{l}\,(\Omega_{\sigma\varepsilon}\, B^{k}_{\gamma} - \Omega_{\sigma\gamma}\, B^{k}_{\varepsilon})\, \dot{u}^{\sigma}.$$

This shows that (2.9) is a sufficient condition for the equality of induced and intrinsic curvature tensors.

The equation (4.6) will be used in finding a relation between the curvature invariant H_2 defined by (3.2) and the curvature tensors of F_n and F_{n-1}. In order to obtain this relation we multiply (4.6) by $g^{\beta\varepsilon} g^{\alpha\gamma}$. The left hand side reduces to

$$\text{(4.7)} \qquad g^{\beta\varepsilon}\, g^{\alpha\gamma}\, K_{\alpha\beta\gamma\varepsilon} = g^{\alpha\gamma}\, K_{\alpha\gamma} = \overline{K},$$

where $\overline{K}$ and $K_{\alpha\gamma}$ are respectively the scalar curvature and Ricci tensor (both intrinsic and induced) of the hypersurface. For simplifying the right hand side we shall use the relations

$$\text{(4.8)} \qquad g^{\beta\varepsilon}\, B^{j}_{\beta}\, B^{k}_{\varepsilon} = g^{jk} - N^{j}\, N^{k}\,,$$

and

$$\text{(4.9)} \qquad K_{jihk} = - K_{ijhk} - 2\, C_{ijl}\, K^{l}_{\gamma hk}\, \dot{x}^{\gamma}$$

(Rund [2] page 105).

The above two equations the condition (2.9) and the skew-symmetry of K_{ijhk} in h and k yield

$$\text{(4.10)} \qquad (K_{ijhk}\, B^{i}_{\alpha}\, B^{j}_{\beta}\, B^{h}_{\gamma}\, B^{k}_{\varepsilon})\, g^{\alpha\gamma}\, g^{\beta\varepsilon} = (K_{ih} - K_{ijhk}\, N^{j}\, N^{k})\,(g^{ih} - N^{i}\, N^{h}) = \\ = K - 2\, K_{ih}\, N^{i}\, N^{h},$$

where K and K_{ih} stand for the scalar curvature and Ricci tensor of F_n. Further, in view of equations (2.12), (3.3) and (3.6) we have

$$\text{(4.11)} \qquad g^{\beta\varepsilon}\, g^{\alpha\gamma}\,(\Omega_{\alpha\gamma}\,\Omega_{\beta\varepsilon} - \Omega_{\alpha\varepsilon}\,\Omega_{\beta\gamma}) = M_1^2 - M_2 = 2\, H_2\,.$$

The last term in (4.6) on being multiplied by $g^{\beta\varepsilon} g^{\alpha\gamma}$ gives

$$\text{(4.12)} \qquad g_{ij}\, B^{j}_{\beta}\, B^{h}_{\alpha}\, \frac{\partial \Gamma^{*i}_{hk}}{\partial \dot{x}^{l}}\, N^{l}\,(\Omega_{\sigma\varepsilon}\, B^{k}_{\gamma} - \Omega_{\sigma\gamma}\, B^{k}_{\varepsilon})\, g^{\beta\varepsilon}\, g^{\alpha\gamma}\, \dot{u}^{\sigma} = \\ = \left(\frac{\partial \Gamma^{*}_{jhk}}{\partial \dot{x}^{l}}\, g^{jk} - \frac{\partial \Gamma^{*}_{jhk}}{\partial \dot{x}^{l}}\, N^{j}\, N^{k} - \frac{\partial \Gamma^{*i}_{hi}}{\partial \dot{x}^{l}} + \frac{\partial \Gamma^{*}_{hjk}}{\partial \dot{x}^{l}}\, N^{j}\, N^{k} \right) N^{l}\, \Omega^{\alpha}_{\sigma}\, B^{h}_{\alpha}\, \dot{u}^{\sigma}\,,$$

where we have used the condition (2.9), the equation (4.8) and the relations

$$B_i^\varepsilon = g_{ij} g^{\beta\varepsilon} B_\beta^j \quad , \quad B_i^\varepsilon B_\varepsilon^k = (\delta_i^k - N^k N_i)$$

in simplification. After differentiating $g^{jk} g_{ij} = \delta_i^k$ and using (2.9) we deduce

$$\frac{\partial g^{jk}}{\partial \dot{x}^l} N^l = 0 .$$

This relation helps in getting

$$\frac{\partial \Gamma_{hi}^{*i}}{\partial \dot{x}^l} N^l = g^{jk} \frac{\partial \Gamma_{hjk}^{*}}{\partial \dot{x}^l} N^l . \tag{4.13}$$

Combining (4.7), (4.10), (4.11), (4.12) and (4.13) we obtain

$$\overline{K} = K - 2 K_{ij} N^i N^j + 2 H_2 + 2 \frac{\partial \Gamma_{[jh]k}^{*}}{\partial \dot{x}^l} (g^{jk} - N^j N^k) N^l \Omega_\sigma^\alpha B_\alpha^h \dot{u}^\sigma \tag{4.14}$$

where

$$\frac{\partial \Gamma_{[jh]k}^{*}}{\partial \dot{x}^l} = \frac{1}{2} \left(\frac{\partial \Gamma_{jhk}^{*}}{\partial \dot{x}^l} - \frac{\partial \Gamma_{hjk}^{*}}{\partial \dot{x}^l} \right) .$$

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