Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Ralph H. F. Denniston

Some packings of projective spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **52** (1972), n.1, p. 36–40.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_52_1_36_0>

Geometria. — *Some packings of projective spaces.* Nota di RALPH H. F. DENNISTON, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Nella geometria proiettiva tridimensionale PG $(3, q)$ sul campo di Galois di ordine q, una *fibrazione* [5] è un insieme di rette dello spazio tale che per ogni punto di questo passi una ed una sola retta dell'insieme. Si dice *riempimento* (packing) un insieme di fibrazioni, tale che ogni retta appartenga ad una ed una sola fibrazione del riempimento. Nella presente Nota si dimostra l'esistenza di riempimenti in una qualsiasi PG $(3, q)$.

In a three-dimensional projective geometry PG $(3, q)$ over a Galois field of order q, a *spread* is a set $\mathcal{S}$ of lines, such that each point lies on one and only one line of $\mathcal{S}$. A *packing* is a set $\{\mathcal{S}\}$ of spreads, such that each line belongs to one and only one spread of $\{\mathcal{S}\}$. Segre [5] and Bruck [1] have suggested that a finite projective plane, of an order which is not a power of a prime, can perhaps be constructed with the help of spreads and packings in some three-dimensional geometry. The interest of this suggestion makes it appear desirable that packings should be constructed in as many ways as possible. But Dembowski remarks [3, page 71, footnote 1] that hardly anything has been published on the subject, apart from the classical case where $q = 2$.

The present paper gives an effective construction for a packing of PG $(3, q)$, where $q > 2$. It can be seen that the construction works equally well in the real projective space; and it could probably be applied to spaces over other infinite (commutative) fields, provided that they were not algebraically closed. It breaks down when $q = 2$, because a ruled quadric uses up more than half the points of PG $(3, 2)$. However, this is just the case about which everything is already known; and the conclusion is that *packings exist in all the finite three-dimensional projective spaces.*

Two types of spread are involved: one is the elliptic linear congruence, which (from the present point of view) is usually referred to as a *regular spread.* Then, if $\mathfrak{R}$ is any regulus contained in a regular spread $\mathcal{S}$, and $\mathfrak{R}^*$ is the complementary regulus, we obtain a spread $\mathcal{S}^*$ by "switching"—taking the lines of $\mathfrak{R}^*$, together with those lines of $\mathcal{S}$ that do not belong to $\mathfrak{R}$. Using the terminology set up by Bruck [1], we may call $\mathcal{S}^*$ "a subregular spread of index 1". In the packing constructed below, the spreads are subregular of index 1, all but one which is regular.

We naturally use the Klein representation of lines in PG $(3, q)$ by points of a quadric primal Ω in PG $(5, q)$. The representation has, at least when q is odd, all the properties which are familiar in classical projective geometry.

(*) Nella seduta del 15 gennaio 1972.

When q is a power of 2, some things are different, because the points of a quadric correspond in a null-polarity to the tangent primes. However, it seems (though I cannot find it explicitly stated anywhere) that all the theorems we need remain valid, even in this case.

PROBLEM 1. *Given a ruled quadric surface* φ *in* PG $(3, q)$, *where* $q > 2$: *to find another ruled quadric, in the same space, having no point in common with* φ.

Construction. Let $\{r\}$ and $\{s\}$ be the two systems of generators on φ. These are represented on Ω by proper conics c and d, in planes ρ and σ (which are mutually polar planes with respect to Ω). Any proper conic in PG $(2, q)$ has exterior lines [3, page 148]; choose lines x and y, in ρ and σ, which do not meet c and d. Then the polar 3-space of x will contain σ, and will intersect Ω in a quadric surface δ; since x does not meet Ω, δ represents an elliptic linear congruence, and is a non-ruled quadric with $q^2 + 1$ points [3, page 48]. Two of these are points of contact of tangent planes that go through y, and another $q + 1$ are points of d; but, since $q > 2$, there are points of δ with neither of these properties. Join one such point to y by a plane τ; then τ must meet δ in a proper conic e, representing a regulus $\{t\}$. The required quadric ψ is generated by $\{t\}$.

Proof. Suppose if possible that φ and ψ have a common point: then there are lines, belonging to $\{s\}$ and $\{t\}$, which either coincide or meet. These are represented by points of d and e which either coincide, or are joined by a line lying on Ω. But the first case is impossible, because τ and d have no common points: and so is the second, because δ is a non-ruled quadric. Therefore ψ is a ruled quadric disjoint from φ, which was required to be found.

PROBLEM 2. *To construct a packing of spreads in* PG $(3, q)$, *where* q *is any prime power greater than* 2.

Construction. Choose a point D of Ω, representing a line d, and let ∇ be the tangent prime to Ω at D. The section of Ω by ∇ is a point-cone Δ, projecting a ruled quadric surface, and represents the lines that meet d. Choose a 3-space Θ lying in ∇, but not going through D. Choose a ruled quadric surface φ lying in Θ, but disjoint from the quadric in which Θ cuts Δ; this is possible, since we have solved Problem 1. Join φ to D by a point-cone Φ, which will have no point in common with Ω except D. Choose two generators m and m', of the same system on φ, and join m' to D by a plane μ.

All these objects being fixed, let us consider a variable point T_i of m; T_i is projectively related to the point U_i of m' such that T_iU_i is a generator of φ, and also to the line DU_i (in the pencil of lines through D in μ). Consider a variable point V_{ij} of DU_i, distinct from D: join V_{ij} to T_i by a line s_{ij}, and to m by a plane ρ_{ij}. Then s_{ij} lies in the point-cone Φ, and does not meet Ω. The section of Ω by ρ_{ij} includes more points than one (because Δ projects a ruled quadric surface), but contains no line (which would meet m), and is therefore a proper conic. Let $\mathcal{S}_{ij}$, $\mathfrak{R}_{ij}$, $\mathfrak{R}^*_{ij}$ be the sets of lines, in

PG $(3, q)$, corresponding to the sections of Ω by the polar spaces of s_{ij} and of ρ_{ij}, and by ρ_{ij} itself. Then $\mathcal{S}_{ij}$ is an elliptic linear congruence; $\mathcal{R}_{ij}$ is a regulus contained in $\mathcal{S}_{ij}$ and including d (ρ_{ij} goes through s_{ij} in ∇), and $\mathcal{R}_{ij}^*$ is the complementary regulus. Replacing $\mathcal{R}_{ij}$ by $\mathcal{R}_{ij}^*$, we can switch the regular spread $\mathcal{S}_{ij}$ into $\mathcal{S}_{ij}^*$, a subregular spread of index 1. Doing this for every point T_i of m, and every point V_{ij} (not D) of the line DU_i corresponding to T_i, we get a collection $\{\mathcal{S}_{ij}^*\}$ of subregular spreads. Finally, the fixed line m does not meet Ω, and the section by its polar space represents a regular spread $\mathfrak{M}$. The required packing consists of the set $\{\mathcal{S}_{ij}^*\}$ of spreads ($i = 0, \cdots, q$; $j = 1, \cdots, q$, say), together with the one spread $\mathfrak{M}$.

Proof. All we have to establish is that, if g is any line of PG $(3, q)$, there is one and only one spread (of the collection we have chosen) to which g belongs. Let g be represented on Ω by a point G, and let Γ be the tangent prime at G to Ω.

First, let g be d. The 4–space ∇ contains m and every ρ_{ij}, so d belongs to $\mathfrak{M}$ and to every $\mathcal{R}_{ij}$. The effect of the switching is to exclude d from every $\mathcal{S}_{ij}^*$, and leave it belonging to $\mathfrak{M}$ alone.

Secondly, let g meet d: then (since lines of a spread do not meet) g cannot belong to $\mathfrak{M}$ nor to any $\mathcal{S}_{ij}$. Also G, which lies on Δ, is not on the plane joining m to D (this plane being contained in Φ). Join G to m by a plane π: then (in the 4–space ∇) π will meet the plane μ (skew to m) in a single point V_{ij}, distinct from D. There will be a point T_i of m to correspond to DV_{ij} in the projectivity; and so we identify π with one of the planes ρ_{ij}—the only plane ρ_{ij} to which G can belong. We then have g belonging to a unique $\mathcal{R}_{ij}^*$, and so to a unique $\mathcal{S}_{ij}^*$.

Thirdly, let g be skew to d and belong to $\mathfrak{M}$: then g also belongs to more than one spread $\mathcal{S}_{ij}$. But, for any such $\mathcal{S}_{ij}$, the prime Γ, since it contains the lines m and s_{ij}, will contain the plane ρ_{ij} that joins them; and so g will belong to $\mathcal{R}_{ij}$, and be excluded from $\mathcal{S}_{ij}^*$. Nor can g belong to any $\mathcal{R}_{ij}^*$; for G is now outside the 4–space ∇, in which every plane ρ_{ij} lies. So g belongs only to $\mathfrak{M}$.

Fourthly, let g be skew to d, without belonging to $\mathfrak{M}$: then the prime Γ does not contain m, but meets m in one point T_i. The corresponding line DU_i meets Γ (which does not go through D) in one point V_{ij} distinct from D. So it happens just once that Γ goes through s_{ij} (but not through ρ_{ij}), and that g belongs to $\mathcal{S}_{ij}$ without belonging to $\mathcal{R}_{ij}$, and accordingly belongs to $\mathcal{S}_{ij}^*$. As in the third case, g cannot belong to any $\mathcal{R}_{ij}^*$.

This completes the proof. The subregular spreads $\mathcal{S}_{ij}^*$, together with the regular spread $\mathfrak{M}$, make up the packing which was to be constructed.

So it is established that every PG $(3, q)$ has a packing. One corollary is that, if q is a prime power, $q^3 + q^2 + q + 1$ schoolgirls could walk in $q^2 + 1$ rows of $q + 1$ each, every day for $q^2 + q + 1$ days, any two girls having one day when they were in the same row.

I add some remarks on the case where $q = 2$; Dembowski (in the footnote cited above) gives many references, but is unnecessarily vague about the results that have been achieved. In fact, Cole [2] tells us that the packings of PG (3 , 2) fall into two transitivity classes under the collineation group. Each class consists of 240 packings, and any packing is invariant under a group of 168 collineations; a point (if the packing belongs to the one class), or a plane (for the other) is left fixed by the collineations of such a group. But the concepts " spread " and " packing " are self-dual in a finite space: so it must be (and I have verified) that a correlation will interchange packings of the two classes. The collineations and correlations of PG (3 , 2) act, on the set of lines, as a permutation group which is transitive on the packings.

Likewise, if $\mathcal{S}$ is a packing constructed by the method of Problems 1 and 2, and $\mathcal{S}^*$ is the image of $\mathcal{S}$ under a correlation, there is no reason to expect that $\mathcal{S}^*$ can be transformed into $\mathcal{S}$ by a collineation. In fact, in Problem 1, $\{t\}$ is related asymmetrically to $\{r\}$ and $\{s\}$. In Problem 2, on the quadric in which Θ cuts Δ, one system of generators is related to points of d, and the other to planes through d; and we have to choose one system on φ as including m and m'. I have verified, in the case where $q = 3$, that $\mathcal{S}$ and $\mathcal{S}^*$ have different projective properties.

In Problem 2, it may be possible to find two points on m, say T_0 and T_1, such that the lines T_0U_1 and T_1U_0 do not meet Ω. This will enable us to use, instead of the projectivity $\{T_i \to U_i\}$, a one-one correspondence $\{T_i \to U_i'\}$, where U_0' is U_1, U_1' is U_0, and U_i' is U_i otherwise. The rest of the construction, carried through as before, will yield a new packing of PG $(3, q)$. If $q > 3$, $\{T_i \to U_i'\}$ is not a projectivity, and the new packing must be projectively different from the old one. This device actually works when $q = 3$, but has then the same effect as the change from $\mathcal{S}$ to $\mathcal{S}^*$ in the last paragraph. It fails when $q = 4$, for special geometrical reasons; but it works when $q = 8$, and presumably for other values of q.

Moreover, when $q > 4$, the solution given for Problem 1 has a modulus, namely the cross-ratio made by σ and τ with the (unordered) pair of tangent planes from y to δ. Different values of this modulus will lead to projectively distinct solutions to Problem 2. So we see that *the collineation group of* PG $(3, q)$ *is not transitive on packings*, except possibly when $q = 4$.

Rao [4] has proposed the problem: Given, in PG $(3, q)$, a collineation T of period $q^2 + q + 1$; to find a spread $\mathcal{S}$ which, together with its own successive images $T\mathcal{S}$, $T^2\mathcal{S}, \cdots$, will make up a packing. It seems very unlikely that this problem can have a solution when $q^2 + q + 1$ is not a prime, because then we can expect the line-orbits of T to vary in length. When $q = 2$, we see from [2] that a solution to the problem can be found in either of the two transitivity classes mentioned above. However, when $q = 3$, there is no solution, as I have found by making an exhaustive search. I hope to discuss in another note the case where $q = 8$, in which there are various solutions to Rao's problem: in fact, $\mathcal{S}$ may in this case be regular, or subregular of index 1, or again $\mathcal{S}$ may contain no regulus at all.

References

[1] R. H. Bruck, *Construction problems of finite projective planes*, « Proc. 1967 Conf. Comb. Math. », Univ. of N. Carolina, Chapel Hill, 426–514 (1969).

[2] F. N. Cole, *Kirkman parades*, « Bull. Amer. Math. Soc. », *28*, 435–437 (1922).

[3] P. Dembowski, *Finite Geometries*, Springer, Berlin–Heidelberg–New York 1968.

[4] C. R. Rao, *Cyclical generation of linear subspaces in finite geometries*, « Proc. 1967 Conf. Comb. Math. », Univ. of N. Carolina, Chapel Hill, 515–535 (1969).

[5] B. Segre, *Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane*, « Ann. di Mat. », (4) *64*, 1–76 (1964).