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Further results on the Boundedness and the Stability of certain fourth order differential equations

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Equazioni differenziali. — *Further results on the Boundedness and the Stability of certain fourth order differential equations.* Nota di H. O. TẸJUMỌLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano due Teoremi sulla limitatezza e stabilità delle soluzioni di una classe di equazioni differenziali non lineari del quarto ordine.

1. INTRODUCTION

Consider the fourth order differential equation

$$x^{(4)} + \varphi_1(\dddot{x})\,\dddot{x} + a_2\,\ddot{x} + \varphi_3(\dot{x}) + a_4\,x = p(t, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.1}$$

in which $a_2 > 0$, $a_4 > 0$ are constants and the functions φ_1, φ_3 and p which depend only on the arguments displayed are such that $\varphi_1(z)$, $\varphi_1'(y)$, $p(t, x, y, z, u)$ are continuous for all x, y, z, u and t. In the case $p \equiv 0$ in (1.1), Ezeilo [1] showed that if $\varphi_3(0) = 0$ and there are constants $a_1 > 0$, $a_3 > 0$ such that

$$\varphi_3(y)/y \geq a_3 \ (y \neq 0) \quad \text{and} \quad \varphi_1(z) \geq a_1 \qquad \text{for all } z, \tag{1.2}$$

$$\{a_1 a_2 - \varphi_3'(y)\}\,a_3 - a_1 a_4 \varphi_1(z) \geq \Delta_0 \qquad (\text{arbitrary } y \text{ and } z) \tag{1.3}$$

for some constant $\Delta_0 > 0$,

$$\varphi_3'(y) - \varphi_3(y)/y \leq \delta_1 \ (y \neq 0), \tag{1.4}$$

where δ_1 is a constant such that $\delta_1 < 2\,\Delta_0\,a_4\,a_1^{-1}\,a_3^{-2}$,

$$z^{-1}\int_0^z \varphi_1(s)\,ds - \varphi_1(z) \leq \delta_2 \ (z \neq 0), \tag{1.5}$$

where δ_2 is a constant such that $\delta_2 < 2\,\Delta_0\,a_1^{-2}\,a_3^{-1}$, then every solution $x(t)$ of (1.1) satisfies

$$x(t) \to 0, \quad \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0, \quad \dddot{x}(t) \to 0 \qquad \text{as } t \to \infty. \tag{1.6}$$

The conditions (1.2) and (1.3) together with $a_i > 0$, $i = 1, 2, 3, 4$ are suitable generalizations of the Routh–Hurwitz conditions

$$a_i > 0 \quad (i = 1, 2, 3, 4), \qquad (a_1 a_2 - a_3)\,a_3 - a_1^2 a_4 > 0$$

(*) Nella seduta dell'11 dicembre 1971.

for the asymptotic stability (in the large) of the trivial solution of the linear equation

$$x^{(4)} + a_1 \dddot{x} + a_1 \ddot{x} + a_3 \dot{x} + a_4 x = 0 .$$

For the general case $p \not\equiv 0$, the present author [2] showed that if, in addition to the above conditions, p is a bounded function, then every solution of (1.1) is ultimately bounded.

The main object of the present note is to extend these results to equations of the form

$$(1.7) \qquad x^{(4)} + \varphi_1(\ddot{x})\, \dddot{x} + \varphi_2(\ddot{x}) + \varphi_3(\dot{x}) + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

of which (1.1) is a special case corresponding to $\varphi_2(\ddot{x})$ linear in (1.7). We shall prove here

THEOREM 1. *Suppose, given the equation*

$$(1.8) \qquad x^{(4)} + \varphi_1(\ddot{x})\, \dddot{x} + \varphi_2(\ddot{x}) + \varphi_3(\dot{x}) + a_4 x = 0 ,$$

that

(i) $\varphi_2(0) = 0 = \varphi_3(0)$,

(ii) *there are constants* $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ *such that each of* (1.2), (1.3), (1.4), (1.5) *and the following hold*:

$$(1.9) \qquad 0 \leq (\varphi_2(z)/z - a_2) \leq \varepsilon_0\, a_3^3/a_4^2 \; (z \neq 0) ,$$

where ε_0 *is a positive constant such that*

$$(1.10) \qquad \varepsilon_0 < \varepsilon \leq \left[\frac{a_4}{a_3} , \frac{\Delta_0}{a_1 a_3 \Delta_1} , \frac{a_1}{4 \Delta_1} \left(\frac{2 \Delta_0}{a_1^2 a_3} - \delta_2 \right) , \frac{a_3}{4 a_4 \Delta_1} \left(\frac{2 a_4 \Delta_0}{a_1 a_3^2} - \delta_1 \right) \right] ,$$

$\Delta_1 = a_1 a_2 + a_2 a_3 a_4^{-1}$. *Then every solution* $x(t)$ *of* (1.8) *satisfies* (1.6).

For the case $p \not\equiv 0$, we shall prove

THEOREM 2. *Let hypotheses* (i) *and* (ii) *of Theorem* 1 *hold and suppose further that*

$$(1.11) \qquad |p(t, x, y, z, u)| \leq A < \infty$$

for all values of t, x, y, z *and* u. *Then there exists a finite constant* $D > 0$ *whose magnitude depends only on* φ_1, φ_2 *and* φ_3 *as well as on* $a_1, a_2, a_3, a_4, \delta_1, \delta_2, \Delta_0$ *and* A *such that every solution* $x(t)$ *of* (1.7) *satisfies*

$$(1.12) \qquad |x(t)| \leq D , \quad |\dot{x}(t)| \leq D , \quad |\ddot{x}(t)| \leq D , \quad |\dddot{x}(t)| \leq D$$

for all $t \geq t_0$ $(0 < t_0 < \infty)$.

A generalization of the Theorem will be given in § 5.

2. Some Preliminaries

A function $W_0(x, y, z, u)$. The main tool in the proofs of Theorems 1 and 2 is the differentiable function $W_0 = W_0(x, y, z, u)$ defined by

$$(2.1)\qquad 2W_0 = a_4 d_2 x^2 + (a_2 d_2 - a_4 d_1) y^2 + 2\int_0^y \varphi_3(\eta)\, d\eta + 2\int_0^z \eta\varphi_1(\eta)\, d\eta$$
$$+ 2\int_0^z \{d_1 \varphi_2(\eta) - d_2 \eta\}\, d\eta + d_1 u^2 + 2a_4 xy + 2a_4 d_1 xz$$
$$+ 2d_1 z\varphi_3(y) + 2d_2 yu + 2zu + 2d_2 y\int_0^z \varphi_1(\eta)\, d\eta$$

where

$$(2.2)\qquad d_1 = \varepsilon + a_1^{-1}, \quad d_2 = \varepsilon + a_1^{-3} a_4,$$

$\varepsilon > 0$ being the constant in (1.10).

Notations. In what follows the capitals $D, D_0, D_1, \cdots$ denote finite positive constants whose magnitudes depend only on the functions $\varphi_1, \varphi_2, \varphi_3$ and p as well as on the constants $a_1, a_2, a_3, a_4, \delta_1, \delta_2, \Delta_0, \varepsilon_0$, and ε, but are independent of solutions of whatever differential equation under consideration. The D_i's, $i = 0, 1, 2$ retain a fixed identity throughout, but the D's without suffixes attached are not necessarily the same each time they occur.

3. Proof of Theorem 1

The procedure here is the same as in [1] and we shall only sketch the outline. Consider, instead of (1.8), the equivalent system

$$(3.1)\qquad \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = u, \quad \dot{u} = -\varphi_1(z)u - \varphi_2(z) - \varphi_3(y) - a_4 x$$

derived from it by setting $y = \dot{x}$, $z = \ddot{x}$ and $u = \dddot{x}$. The whole idea of the proof of the theorem is to show that W_0 is a Lyapunov function for the system (3.1). In fact, we shall verify that

Lemma 1. *Subject to the conditions of Theorem* 1:

(i) $W_0(0, 0, 0, 0) = 0$ *and there exist constants* D_i, $i = 1, 2, 3, 4$ *such that*

$$(3.2)\qquad W_0 \geq D_1 x^2 + D_2 y^2 + D_3 z^2 + D_4 u^2$$

for all x, y, z *and* u;

(ii) *the derivative* $\dot{W}_0 \equiv \dot{W}_0(x(t), y(t), z(t), u(t))$ *corresponding to any solution* $(x(t), y(t), z(t), u(t))$ *of* (3.1) *satisfies*

$$(3.3)\qquad \dot{W}_0 \leq -D_5(y^2 + z^2 + u^2)$$

for some constant D_5.

The usual Barbašin-type argument applied to (3.2) and (3.3) would then show that, for any solution $(x(t), y(t), z(t), u(t))$ of (3.1),

$$x(t) \to 0, \quad y(t) \to 0, \quad z(t) \to 0, \quad u(t) \to 0 \quad \text{as} \quad t \to \infty,$$

which is (1.6).

Proof of Lemma 1. Since $\varphi_2(0) = 0$ and $\varphi_2(z)/z \geq a_2 \ (z \neq 0)$, by (1.8), it is clear from (2.1) that

$$2\,W_0 \geq a_4\, d_2\, x^2 + (a_2\, d_2 - a_4\, d_1)\, y^2 + 2\int_0^y \varphi_3(\eta)\, d\eta + 2\int_0^z \eta\varphi_1(\eta)\, d\eta$$

$$+ (a_2\, d_1 - d_2)\, z^2 + d_1\, u^2 + 2\, a_4\, xy + 2\, a_4\, d_1\, xz +$$

$$+ 2\, d_1\, z\varphi_3(y) + 2\, d_2\, yu + 2\, zu + 2\, d_2\, y\int_0^z \varphi_1(\eta)\, d\eta$$

$$\equiv 2\, \dot{W}_0^*.$$

The function W_0^* is the same as the function V (3.1) of [1] except that here we have φ_1, φ_3 in place of f and g respectively and u in place of w. It will be seen from the various estimates arising in the course of the proof of [1; Lemma 1] that if ε is fixed by (1.10) then W_0^*, and hence W_0, satisfies (3.2).

Turning now to (3.3), let $(x(t), y(t), z(t), u(t))$ be any solution of (3.1). By a straightforward differentiation from (2.1) we have that

$$\dot{W}_0 = -U_1 - U_2 - U_3 - U_4 \tag{3.4}$$

where

$$U_1 = d_2\, y\varphi_3(y) - a_4\, y^2, \quad U_2 = \{a_2 - d_1\, \varphi_3'(y)\}\, z^2 - d_2\, z\int_0^z \varphi_1(\eta)\, d\eta, \tag{3.5}$$

$$U_3 = (d_1\, \varphi_1(z) - 1)\, u^2, \quad U_4 = z\varphi_2(z) - a_2\, z^2 + d_2\,(y\varphi_2(z) - a_2\, yz).$$

By reasoning as in the proof of [1; Lemma 2], it can be shown from (1.2), (1.3) and (1.10) that

$$U_1 \geq a_3\, \varepsilon y^2, \quad U_2 \geq \frac{1}{2}(\Delta_0/a_1\, a_3)\, z^2, \quad U_3 \geq a_1\, \varepsilon u^2. \tag{3.6}$$

Concerning the term U_4, note that if $z \neq 0$,

$$U_4 = (\varphi_2(z)/z - a_2)\,(z^2 + d_2\, yz)$$

$$\geq -(\varphi_2(z)/z - a_2)\,\frac{d_2^2}{4}\, y^2$$

by (1.9). Since $U_4 = 0$ when $z = 0$, U_4 satisfies

$$U_4 \geq -(\varphi_2(z)/z - a_2)\,\frac{d_2^2}{4}\, y^2 \tag{3.7}$$

always. But by (2.2) and (1.9),

$$\frac{1}{4}\, d_2^2\, (\varphi_2(z)/z - a_2) < \varepsilon_0\, a_3\, ;$$

therefore

$$U_1 + U_4 \geq (\varepsilon - \varepsilon_0)\, a_3\, y^2. \tag{3.8}$$

On combining (3.8) and the estimates for U_2 and U_3 in (3.6) with (3.4), we get

$$\dot{W}_0 \leq -(\varepsilon - \varepsilon_0)\, a_3\, y^2 - \frac{1}{2}\, (\Delta_0/a_1\, a_3)\, z^2 - \varepsilon a_1\, u^2,$$

which verifies (3.3). Theorem 1 now follows as was pointed out.

4. Proof of Theorem 2

Here also we consider the differential system

$$\begin{aligned} &\dot{x} = y\,, \quad \dot{y} = z\,, \quad \dot{z} = u\,, \\ &\dot{u} = -\varphi_1(z)\, u - \varphi_2(z) - \varphi_3(y) - a_4\, x + p\,(t\,, x\,, y\,, z\,, u)\,, \end{aligned} \tag{4.1}$$

which is derived from (1.7) on setting $y = \dot{x}$, $z = \ddot{x}$ and $u = \dddot{x}$. Our procedure is the same as in the proof of the analogous result [2; Theorem 1], and we shall prove here that

LEMMA 2. *Assume that the conditions of Theorem 2 hold. Let $W_1 = W_1(x\,, u)$ be the continuous function defined by*

$$W_1 = \begin{cases} x \operatorname{sgn} u\,, & \text{if } |u| \geq |x| \\ u \operatorname{sgn} x\,, & \text{if } |u| \leq |x| \end{cases} \tag{4.2}$$

and set

$$W = W_0 + W_1\,, \tag{4.3}$$

where W_0 is the function (2.1). *Then*

$$W(x\,, y\,, z\,, u) \to +\infty \quad \text{as} \quad x^2 + y^2 + z^2 + u^2 \to \infty, \tag{4.4}$$

and the limit

$$\dot{W}^+ = \limsup_{h \to +0} \{W(t+h)\,, y(t+h)\,, z(t+h)\,, u(t+h) - W(x(t)\,, y(t)\,, z(t)\,, u(t))\}/h$$

exists, corresponding to any solution $(x(t)\,, y(t)\,, z(t)\,, u(t))$ of (4.1), *and satisfies*

$$\dot{W}^+ \leq -D_5 \quad \textit{provided} \quad x^2(t) + y^2(t) + z^2(t) + u^2(t) \geq D_6 \tag{4.5}$$

for some constants $D_5\,, D_6$.

As shown in [2; § 4], the two results (4.4) and (4.5) imply that

$$x^2(t) + y^2(t) + z^2(t) + u^2(t) \leq D, \qquad t \geq t_0 \quad (0 < t_0 < \infty),$$

which is precisely (1.12).

Proof of Lemma 2. Clearly, from (4.2),

$$|W_1| \leq |u|$$

for all x and u, so that by (4.3) and (3.2)

$$W \geq D_1 x^2 + D_2 y^2 + D_3 z^2 + D_4 u^2 - |u|,$$

from which (4.4) follows.

Next we verify (4.5). Let $(x(t), y(t), z(t), u(t))$ be any solution of (4.1). Then

$$\dot{W}_1^+ = \limsup_{h \to +0} \{W_1(x(t+h), u(t+h)) - W_1(x(t), u(t))\}/h,$$

and a straightforward calculation from (4.2) and (4.1) gives that

$$\dot{W}_1^+ = \begin{cases} y \operatorname{sgn} u, & \text{if } |u| \geq |x| \\ -(\varphi_1(z)u + \varphi_2(z) + \varphi_3(y) + a_4 x - p) \operatorname{sgn} x, & \text{if } |u| \leq |x| \end{cases}.$$

Thus, by (1.9) and (1.11),

$$(4.6) \quad \dot{W}_1^+ \leq \begin{cases} |y|, & \text{if } |u| \geq |x| \\ -a_4|x| + |\varphi_3(y)| + D_7(1 + |z| + |u|), & \text{if } |u| \leq |x| \end{cases},$$

where in obtaining (4.6) we also used the fact, arising from (1.3), that $\varphi_1(z) < a_2 a_3 a_4^{-1}$ for all z.

Observe now from (2.1) and (4.1) that

$$\dot{W}_0 = -U_1 - U_2 - U_3 - U_4 + (d_2 y + z + d_1 u)\, p(t, x, y, z, u)$$

U_1, U_2, U_3 and U_4 being as given in (3.5); therefore, by (1.11),

$$\dot{W}_0 \leq -U_1 - U_2 - U_3 - U_4 + D_8(|y| + |z| + |u|)$$

where $D_8 = \max(1, d_1, d_2)A$. From this, (3.6) and (4.6) it is clear that $\dot{W}^+ = \dot{W}_0 + \dot{W}_1^+$ necessarily satisfies

$$(4.7) \quad \dot{W}^+ \leq -U_1 - U_4 - \tfrac{1}{2}(\Delta_0/a_1 a_3) z^2 - a_1 \varepsilon u^2 + D_9(|y| + |z| + |u|)$$

if $|u| \geq |x|$, or

$$(4.8) \quad \begin{aligned} \dot{W}^+ \leq &-(U_1 - |\varphi_3(y)|) - U_4 - \tfrac{1}{2}(\Delta_0/a_1 a_3) z^2 - a_1 \varepsilon u^2 - a_4|x| \\ &+ D_{10}(1 + |y| + |z| + |u|) \end{aligned}$$

if $|u| \leq |x|$.

First we show that there is a constant D_{11} such that

$$\dot{W}^{+} \leq -1 \quad \text{whenever} \quad y^2 + z^2 + u^2 \geq D_{11}^2 . \tag{4.9}$$

Indeed let $|y| > d_2^{-1}$. Then, by (3.5), (1.2) and (2.2)

$$\begin{aligned} U_1 - |\varphi_3(y)| &= (d_2|y| - 1)|\varphi_3(y)| - a_4 y^2 \\ &\geq a_3 \varepsilon y^2 - a_3 |y| , \end{aligned}$$

so that, as in the verification of (3.8),

$$(U_1 - |\varphi_3(y)|) + U_4 \geq (\varepsilon - \varepsilon_0) a_3 y^2 - a_3 |y| .$$

On using this in (4.8) and noting that $U_1 + U_4$ in (4.7) satisfies (3.8), it will be clear from (4.7) and (4.8) that $\dot{W}^{+}$, at least, satisfies

$$\dot{W}^{+} \leq -D_{12} y^2 + D(|y| + 1)$$

for some constant D_{12} if $|y| > d_2^{-1}$. Thus, provided $|y| > d_2^{-1}$ is large enough, $|y| > D_{13}$ say,

$$\dot{W}^{+} \leq -1 \quad \text{if} \quad |y| \geq D_{13} \; (> d_2^{-1}) . \tag{4.10}$$

If, however, $|y| \leq d_2^{-1}$ it will be seen from (4.7), (4.8), (3.6) and (3.7) that

$$\dot{W}^{+} \leq -\frac{1}{2} (\Delta_0 / a_1 a_3) z^2 - a_1 \varepsilon u^2 + D(|z| + |u| + 1),$$

from which it follows that $\dot{W}^{+} \leq -1$ when $|y| \leq d_2^{-1}$ provided $z^2 + u^2$ is large enough, say $z^2 + u^2 \geq D_{14}$. In other words,

$$\dot{W}^{+} \leq -1 \quad \text{if} \quad y^2 + z^2 + u^2 \geq D_{12}^2 + D_{14}^2 , \tag{4.11}$$

which is (4.9) with $D_{11}^2 = D_{13}^2 + D_{14}^2$.

Next, we verify that the estimate (4.11) still holds when $y^2 + z^2 + u^2 \leq D_{11}^2$ provided that $|x|$ is large enough. Assume here, to start with, that $|x| \geq D_{11}$. Then $|x| \geq |u|$ so that $\dot{W}^{+}$ satisfies (4.8). Since $y^2 + z^2 + u^2 \leq D_{11}$, it is clear here that

$$\begin{aligned} \dot{W}^{+} &\leq -a_4 |x| + D_{15} \\ &\leq -1 , \end{aligned}$$

provided $|x| \geq D_{11}$ is sufficiently large, say $|x| \geq D_{16}$ ($\geq D_{11}$). Thus

$$\dot{W}^{+} \leq -1 \quad \text{if} \quad y^2 + z^2 + u^2 \leq D_{11}^2 \quad \text{but} \quad |x| \geq D_{16} . \tag{4.12}$$

The result (4.12) combined with (4.11) clearly show that

$$\dot{W}^{+} \leq -1 \quad \text{if} \quad x^2 + y^2 + z^2 + u^2 \geq D_{16}^2 + D_{11}^2 ,$$

which verifies (4.5).

Theorem 2 now follows as was pointed out.

5. A GENERALIZATION OF THEOREM 2

There is no difficulty in extending Theorem 2 to an equation (1.7) in which p satisfies

$$|p(t, x, y, z, u)| \leq A + B(y^2 + z^2 + u^2)^{1/2} \tag{5.1}$$

with $A > 0$ and $B > 0$ constants and B sufficiently small. Indeed the use of (5.1) instead of the condition (1.11) of Theorem 2 does not affect the work in § 4 appreciably. The main adjustments would occur in the two estimates (4.7) and (4.8) of $\dot{W}^+$, each of which will now have to be augmented by a term not exceeding $BD_{17}(y^2 + z^2 + u^2)^{1/2}$, $D_{17} = 3^{1/2} \max(1, d_1, d_2)$. However, if B is fixed such that

$$B < \min\left[\frac{a_3 \varepsilon}{2 D_{17}}, \frac{\Delta_0}{4 a_1 a_3 D_{17}}, \frac{a_1 \varepsilon}{2 D_{17}}\right],$$

it will be seen, by using the arguments of § 4, that the two estimates (4.9) and (4.12) for $\dot{W}^+$ still hold valid under the condition (5.1).

REFERENCES

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[2] H. O. TEJUMOLA, « Ann. Mat. Pura Appl. », (IV) *80*, 177–196 (1968).