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Boundedness Theorems for some systems of two differential equations

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Equazioni differenziali. — *Boundedness Theorems for some systems of two differential equations.* Nota di H. O. TẸJUMỌLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano due Teoremi di definitiva limitatezza per le soluzioni di un sistema di due equazioni differenziali ordinarie del primo ordine, non lineari.

1. INTRODUCTION

The systems of real ordinary differential equations considered here are of the form

$$\dot{x} = h(y), \quad \dot{y} = -f(x, y)\, y - g(x) + p(t, x, y), \tag{1.1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and the functions f, g, h and p depend only on the arguments explicitly displayed in (1.1).

When $h(y) = y$, (1.1) becomes

$$\dot{x} = y, \quad \dot{y} = -f(x, y)\, y - g(x) + p(t, x, y), \tag{1.2}$$

a system which is derivable from the second order equation

$$\ddot{x} + f(x, \dot{x})\, \dot{x} + g(x) = p(t, x, \dot{x}) \tag{1.3}$$

by setting $\dot{x} = y$. A large number of boundedness Theorems (for example see [1], [4], [5]) are already available for equations of the form (1.3) or for systems, such as (1.2), derived from such equations. Very little seems to have been done for general systems of the form (1.1). In particular see [3].

The object of this note is to investigate conditions under which solutions of (1.1) are uniform-ultimately bounded. Our results extend some of the known results ([2], [6] for example) for (1.3).

It will be assumed throughout the sequel that the functions f, g, h and p in (1.1) are continuous. Our first result is as follows.

THEOREM 1. *Let* $a, b, \mathrm{L}, \mathrm{F}$ *and* A *be positive constants such that:*

$$|p(t, x, y)| \leq \mathrm{A} \quad \textit{for all } t, x \textit{ and } y, \tag{1.4}$$

$$|y|\, f(x, y) \geq a \quad (|y| \geq \mathrm{L}), \quad \gamma(x) = \max_{|y| \leq \mathrm{L}} |f(x, y)|, \quad a > \mathrm{A} + 1, \tag{1.5}$$

$$h(y)/y \geq b \quad (|y| \geq \mathrm{L}), \quad \mathrm{F} = \max_{|y| \leq \mathrm{L}} |h(y)|. \tag{1.6}$$

Suppose further that

$$g(x)\, \mathrm{sgn}\, x - \mathrm{L}(\mathrm{LF} + 1)\, \gamma(x) \to +\infty \quad \textit{as} \quad |x| \to \infty. \tag{1.7}$$

(*) Nella seduta dell'11 dicembre 1971.

Then there exists a finite positive constant K *whose magnitude depends only on the constants* a , b , L , F *and* A *as well as on the functions* g *and* γ *such that every solution* $(x(t), y(t))$ *of* (1.1) *ultimately satisfies*

$$|x(t)| \leq K \quad , \quad |y(t)| \leq K. \tag{1.8}$$

The restriction $a > A + 1$ in (1.5) can be dispensed with altogether if f satisfies the stronger condition $f(x, y) \geq a$ $(|y| \geq L)$. It will be seen from our next result that, in this case, the above condition on $h(y)$ can also be considerably relaxed.

THEOREM 2. *Let* a , L , F *and* A *be positive constants such that*:

$$f(x, y) \geq a \quad (|y| \geq L), \quad \gamma(x) = \max_{|y| \leq L} |f(x, y)|, \tag{1.9}$$

$$yh(y) > 0 \quad (|y| \geq L), \quad F = \max_{|y| \leq L} |h(y)|, \tag{1.10}$$

$$|p(t, x, y)| \leq A \qquad \text{for all } t, x \text{ and } y. \tag{1.11}$$

Suppose further that g *satisfies* (1.7) *and that*

$$ayh(y) - (A + 1)|h(y)| \to +\infty \qquad \text{as} \quad |y| \to \infty. \tag{1.12}$$

Then there exists a finite positive constant K *whose magnitude depends only on* a , L , F *and* A *as well as on the functions* g, h *and* γ *such that every solution* $(x(t), y(t))$ *of* (1.1) *ultimately satisfies*

$$|x(t)| \leq K \quad , \quad |y(t)| \leq K.$$

2. PROOF OF THEOREMS 1 AND 2

Our method of proof, which makes use of an adaptation of the well known Yoshizawa technique, is the same as in [2]. We start with the proof of Theorem 1.

Let the continuous function $U = U(x, y)$ be defined by

$$U = U_1 + U_2 + 1, \tag{2.1}$$

where

$$U_1 = \int_0^y h(\eta)\, d\eta + \int_0^x g(\xi)\, d\xi \tag{2.2}$$

and

$$U_2 = \begin{cases} \frac{y}{L} \operatorname{sgn} x \quad , & |y| \leq L \\ \operatorname{sgn} x \operatorname{sgn} y, & |y| \geq L \end{cases} \qquad \text{if} \quad |x| \geq 1 \tag{2.3}$$

or,

$$U_2 = \begin{cases} \frac{xy}{L} \quad , & |y| \leq L \\ x \operatorname{sgn} y, & |y| \geq L \end{cases} \qquad \text{if} \quad |x| \leq 1. \tag{2.4}$$

Since by definition, $|U_2| \leq 1$, (2.1) yields $U \geq U_1$, so that by (2.2) and hypotheses (1.6) and (1.7),

$$U(x, y) \to \infty \qquad \text{as} \quad x^2 + y^2 \to \infty. \tag{2.5}$$

For any solution $(x(t), y(t))$ of (1.1), let

$$\dot{U}^* = \limsup_{h \to +0} \{U(x(t+h), y(t+h)) - U(x(t), y(t))\}/h.$$

U^* exists since $U = U(x, y)$ is at least locally lipschitzian in x and y. We shall show that there are finite constants $K_0 > 0$, $K_1 > 0$ such that

$$\dot{U}^* \leq -K_0 \qquad \text{if} \quad x^2(t) + y^2(t) \geq K_1. \tag{2.6}$$

From this and (2.6) it will then follow, just as in [2; § 5], that there is a constant $K > 0$ such that every solution $(x(t), y(t))$ of (1.1) ultimately satisfies

$$x^2(t) + y^2(t) \leq K,$$

and this verifies (1.8).

To verify (2.6), observe from (2.1) to (2.4) and (1.1) that

$$\dot{U}^* = \dot{U}_1 + \dot{U}_2^*, \tag{2.7}$$

where

$$\dot{U}_1 = -yh(y) f(x, y) + h(y) p(t, x, y) \tag{2.8}$$

and

$$\dot{U}_2^* = \begin{cases} -\frac{1}{L}(f(x, y) y + g(x) - p) \operatorname{sgn} x, & |y| \leq L \\ 0, & |y| \geq L \end{cases} \quad \text{if} \quad |x| \geq 1 \tag{2.9}$$

or,

$$\dot{U}_2^* = \begin{cases} -\frac{x}{L}(f(x, y) y + g(x) - p) + \frac{y}{L} h(y), & |y| \leq L \\ h(y) \operatorname{sgn} y, & |y| \geq L \end{cases} \quad \text{if} \quad |x| \leq 1. \tag{2.10}$$

Thus, if $|y| \leq L$, $\dot{U}_2^*$ satisfies

$$\dot{U}_2^* \leq -\frac{1}{L} g(x) \operatorname{sgn} x + |f(x, y)| + \frac{1}{L}|p| \tag{2.11}$$

or,

$$\dot{U}_2^* \leq -\frac{x}{L} g(x) + |f(x, y)| + \frac{1}{L}|p| \tag{2.12}$$

according as $|x| \geq 1$ or $|x| \leq 1$. But if $|y| \geq L$, then

$$\dot{U}_2^* = \begin{cases} 0, & |x| \geq 1 \\ h(y) \operatorname{sgn} y, & |x| \leq 1 \end{cases}. \tag{2.13}$$

In obtaining estimates for $\dot{U}^*$ we shall consider points outside of the closed bounded set defined by $|x| \leq 1$ and $|y| \leq L$. It will be convenient to consider the following three regions in turn: (I) $|x| \geq 1$ and $|y| \leq L$, (II) $|x| \leq 1$ and $|y| \geq L$, and (III) $|x| \geq 1$ and $|y| \geq L$. For the case (I), we have from (2.7), (2.8) and (2.11) that

$$\dot{U}^* \leq -yh(y) f(x, y) + h(y) p - \frac{1}{L} g(x) \operatorname{sgn} x + |f(x, y)| + \frac{1}{L} |p|$$

so that, by (1.4), (1.5) and (1.6),

$$\dot{U}^* \leq -\frac{1}{L} [g(x) \operatorname{sgn} x - L(LF + 1) \gamma(x)] + \frac{A}{L} (LF + 1), \tag{2.14}$$

since $|y| \leq L$. Thus, in view of (1.7), there exists a finite constant $K_2 (> 1)$, sufficiently large, such that

$$\dot{U}^* \leq -1 \quad \text{provided} \quad |x| \geq K_2. \tag{2.15}$$

As for the case (II): $|x| \leq 1$ and $|y| \geq L$ note first that by (1.6),

$$yh(y) = |y| |h(y)| \quad \text{and} \quad h(y) \operatorname{sgn} y = |h(y)| \qquad \text{for} \quad |y| \geq L.$$

Hence (2.7), (2.8) and (2.13) together show that

$$\dot{U}^* \leq -|y| |h(y)| f(x, y) + (A + 1) |h(y)|$$

and, in view of (1.5),

$$\dot{U}^* \leq -(a - A - 1) |h(y)|.$$

If we set $a_* = a - A - 1$, then, by (1.6),

$$\dot{U}^* \leq -1 \qquad \text{if} \quad |y| \geq \max(L, a_*^{-1} b^{-1}) = K_3. \tag{2.16}$$

The case (III): $|x| \geq 1$ and $|y| \geq L$ follows from case (II) since $\dot{U}_2^* = 0$ if $|x| \geq 1$ and $|y| \geq L$. The two results (2.15) and (2.16) together imply that

$$\dot{U}^* \leq -1 \quad \text{provided} \quad x^2 + y^2 \geq K_2^2 + K_3^2.$$

This verifies (2.6) and Theorem 1 now follows.

The arguments for the proof of Theorem 2 is the same as that for Theorem 1 except for some modifications which we now outline. We make use of the same function $U = U(x, y)$ (2.1) which, in view of (1.7) and (1.12), satisfies (2.5).

Turning now to (2.6), observe that the estimate (2.15) still holds. As for the case (II): $|x| \leq 1$ and $|y| \leq L$, it will be clear from (2.7), (2.8) and (2.13) that

$$\dot{U}^* \leq -yh(y) f(x, y) + h(y)(\operatorname{sgn} y + p),$$

so that by (1.9), (1.10) and (1.11),

$$\dot{U}^* \leq -ayh(y) + (A + 1) |h(y)|.$$

Thus, by (1.12), there exists a constant $K_4 (> L)$, sufficiently large, such that

$$\dot{U}^* \leq -1 \quad \text{if} \quad |y| \geq K_4 .$$

As before, we have that

$$\dot{U}^* \leq -1 \quad \text{provided} \quad x^2 + y^2 \geq K_2^2 + K_4^2$$

and Theorem 2 now follows.

3. A refinement of Theorems 1 and 2

The hypothesis (1.7) may seem rather too restrictive in that it admits only of unbounded functions $g(x)$. There is, however, no difficulty in extending Theorems 1 and 2 to the case where g satisfies, instead of (1.7), the slightly weaker conditions:

$$\varliminf_{|x| \to \infty} \{ g(x) \operatorname{sgn} x - (LF + 1) \gamma(x) \} > A(LF + 1), \tag{3.1}$$

$$\lim_{|x| \to \infty} \int_0^x g(\xi) \, d\xi = + \infty . \tag{3.2}$$

((3.1) and (3.2) allow for bounded as well as unbounded functions $g(x)$). Indeed the use of (3.2) together with (1.6) or (1.12) still shows that $U = U(x, y)$ satisfies (2.5). The need for further use of (1.7) in § 2 arose only in the proof of (2.15). Since (3.1) implies the existence of finite constants $K_5 > 0$, $K_6 > 0$ such that

$$|x| \geq K_5 \Rightarrow g(x) \operatorname{sgn} x - (LF + 1) \gamma(x) - A(LF + 1) > K_6$$

it will be clear from (2.14) that $\dot{U}^*$ satisfies

$$\dot{U}^* \leq - K_6 \quad \text{provided} \quad |x| \geq \max(K_5, 1),$$

and this verifies (2.15). The rest of the arguments in § 2 are unaffected by the use of (3.1) and (3.2).

References

[1] T. A. Burton, « Ann. Mat. Pura. ed. Appl. (IV) », *85*, (1970).

[2] J. O. C. Ezeilo, « Ann. Mat. Pura. ed. Appl. (IV) », *65*, 389–405 (1964).

[3] I. Muntean, « Mathematica », *11* (34), 2, 299–305 (1969).

[4] R. Reissig, G. Sansone and R. Conti, *Qualitative Theorier Nichtlinear Differentialgleichungen*. (Roma 1963).

[5] G. Sansone and R. Conti, *Non-linear Differential Equations*, Pergamon Press, New York.

[6] H. O. Tejumola, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. », ser. VIII, Vol. L, fasc. 4 (April 1971).