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Supports in Product Spaces

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Supports in Product Spaces.* Nota di ROGER H. MARTY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questo lavoro che fa seguito a due Note lincee di altro Autore [1, 2], vengono studiati i sostegni delle funzioni continue definite su prodotti di spazi topologici soddisfacenti a certe condizioni di densità, di grado di cellularità, eccetera.

Recent studies have been made concerning supports of continuous functions (see for example Alas [1], [2], Marty [4], Ross and Stone [5], Ulmer [7], and earlier papers which these reference). The purpose of this note is to give generalizations of some results in [1], [4], and [5].

All spaces are assumed to be Hausdorff and to have at least two elements. The *degree of cellularity*, $c(X)$, of a space X is the supremum of the cardinalities of the families of nonempty mutually disjoint open subsets of X. The *density* of a space X is the smallest cardinal m such that X has a dense subset of cardinality m, and the *weight* of X is the smallest infinite cardinal m such that X has a base of cardinality at most m. For an infinite cardinal p and a collection $\{X_\alpha : \alpha \in A\}$ of spaces we denote by X the Cartesian product of this collection with the *p–topology* (i.e., elementary neighborhoods are of the form $\cap \{\pi_{\alpha_\xi}^{-1}[U_\xi] : \xi \in \Xi\}$ where card $\Xi < p$ and U_ξ is an open set in X_{α_ξ} for every $\xi \in \Xi$). A *support* of a function f defined on $S \subset X$ is a set $B \subset A$ for which $f(x) = f(y)$ for every $x, y \in S$ with $x|B = y|B$ ($x|B$ denotes the restriction of x to B). A *support* of a subset F of X is a set $B \subset A$ for which $x \in F$, $y \in X$, and $x|B = y|B$ imply that $y \in F$ for every $x \in F$, $y \in X$.

THEOREM 1. *If* $c(X) \leq m$, *then every regular closed subset of* X *has a support of cardinality* $\leq \max\{m, p\}$.

Proof. Let $F = \bar{U}$. By Zorn's Lemma U contains a maximal family $\mathcal{O}$ of mutually disjoint elementary neighborhoods. Since $c(X) \leq m$,

(*) Nella seduta dell'11 dicembre 1971.

card $\mathcal{O} \leq m$. For every $G \in \mathcal{O}$ let $F(G)$ denote the set (of cardinality $< p$) of distinguished indices associated with G and let $B = \cup\{F(G) : G \in \mathcal{O}\}$. Then card $B \leq \max\{m, p\}$ and B is a support of $\cup\mathcal{O}$. It follows that B is a support of $\overline{\cup\mathcal{O}}$. By the maximality of $\mathcal{O}$, $\bar{U} = \overline{\cup\mathcal{O}}$.

THEOREM 2. *If the density of* X_α *is* $\leq m^p$ *for every* $\alpha \in A$, *then* $c(X) \leq m^p$.

Proof. Let $\{G_\xi : \xi \in \Xi\}$ be a family of nonempty mutually disjoint open subsets of X. Without loss of generality we may assume that each G_ξ is an elementary neighborhood. Let $\Gamma \subset \Xi$ such that card $\Gamma \leq 2^{m^p}$. If $B = \cup\{F(G_\xi) : \xi \in \Gamma\}$, then card $B \leq 2^{m^p}$. The family $\{G_\xi | B : \xi \in \Gamma\}$ $(G_\xi | B = \{x | B : x \in G_\xi\})$ consists of nonempty mutually disjoint open subsets of $X | B$. By [1, Theorem 1], $X | B$ with the p^*–topology (p^* denotes the cardinal successor of p) has density $\leq m^p$. Thus $X | B$ with the coarser p–topology also has density $\leq m^p$. Consequently, card $\Gamma \leq m^p$. Thus $c(X) \leq m^p$.

COROLLARY 1. *If the density of* X_α *is* $\leq m^p$ *for every* $\alpha \in A$, *then every regular closed subset of* X *has a support of cardinality* m^p.

COROLLARY 2. *Suppose the weight of* X_α *is* $\leq m^p$ *for every* $\alpha \in A$. *Then for every pair* U, V *of disjoint open subsets of* X, *there are disjoint sets* U', V', *each of which can be represented as a union of a collection of cardinality* $\leq m^p$ *of elementary neighborhoods, and such that* $U \subset U'$ *and* $V \subset V'$.

Proof. By Corollary 1, $\bar{U}$ and $\bar{V}$ have supports $B(U)$ and $B(V)$ each having cardinality $\leq m^p$. Let $B = B(U) \cup B(V)$. Then card $B \leq m^p$. Let $U' = \pi_B^{-1}[U | B]$ and $V' = \pi_B^{-1}[V | B]$. Then U' and V' satisfy the thesis since the weight of $X | B$ is $\leq m^p$.

THEOREM 3. *Let f be a continuous function from* X *into a completely regular space* Y *of weight* q.

a) *If every regular closed set in* X *has a support of cardinality* $\leq r$, *then f has a support of cardinality* $\leq \max\{r, q\}$.

b) *If* $c(X) \leq m$, *then f has a support of cardinality* $\leq \max\{m, p, q\}$

c) *If the density of* X_α *is* $\leq m^p$ *for every* $\alpha \in A$, *then f has a support of cardinality* $\leq \max\{m^p, q\}$.

Proof. *a*) Let $Y_0 = f[X]$ and embedd Y_0 in the Tihonov product of the collection $\{I_\xi : \xi \in \Xi\}$ where Ξ is an index set of cardinality q and $I_\xi = I = [0, 1]$ for every $\xi \in \Xi$. For every $\xi \in \Xi$ and every $n \in N$ (the set of positive integers) there is a countable open cover $\mathcal{C}_{\xi,n}$ of $(\pi_\xi \circ f)[X]$ by elements with diameters $< 1/n$. Let $\mathcal{S}_{\xi,n} = \{(\pi_\xi \circ f)^{-1}[C] : C \in \mathcal{C}_{\xi,n}\}$ for every $\xi \in \Xi$, $n \in N$ and let $\mathcal{S} = \cup\{\mathcal{S}_{\xi,n} : \xi \in \Xi, n \in N\}$. Then card $\mathcal{S} \leq q$ and $\mathcal{S}$ consists of open sets. Thus for every $U \in \mathcal{S}$ there is support $B(U)$ of $\bar{U}$ of cardinality $\leq r$. The set $B = \cup\{B(U) : U \in \mathcal{S}\}$ is a support of f of cardinality $\leq \max\{r, q\}$.

b) This follows from Theorem 1 and part *a*) above.

c) This follows from Theorem 2 and part *b*) above.

THEOREM 4. *If* $c(X_\alpha) \leq \aleph_0$ *for every* $\alpha \in A$ *and* X *has the Tihonov topology, then every continuous function f from* X *into a completely regular space of weight* q *has a support of cardinality* q.

Proof. This follows from Theorem 3 *a*) and a result of Sparks [6] that a regular open set in X has a countable support.

COROLLARY 3. *If* $c(X_\alpha) \leq \aleph_0$ *for every* $\alpha \in A$ *and* X *has the Tihonov topology, then every continuous function from* X *into a separable metric space has a countable support.*

The following theorem is an application of Theorem 3.

THEOREM 5. *If* X_α *is a discrete space of cardinality* m^p *for every* $\alpha \in A$ *and* X *is not discrete* (i.e., card $A > m^p$), *then* X *is not normal.*

Proof. Without loss of generality we may assume that each X_α consists of the same set of elements. Fix $a, b \in X_\alpha$ and let $F_a = \{x \in X :$ for $c \neq a$, there is at most one $\alpha \in A$ with $x(\alpha) = c\}$ and $F_b = \{x \in X :$ for $c \neq b$, there is at most one $\alpha \in A$ with $x(\alpha) = c\}$. It is easy to check that F_a and F_b are disjoint and closed.

If X is normal, then by Urysohn's Lemma there is a continuous $f : X \rightarrow [0, 1]$ such that $f[F_a] = \{0\}$ and $f[F_b] = \{1\}$. By Theorem 3 *c*), f has a support B of cardinality m^p. Let φ be an injection of B into X_α and define $y, y' \in X$ such that $y(\alpha) = \varphi(\alpha) = y'(\alpha)$ for every $\alpha \in B$, and for $\alpha \in A - B$ let $y(\alpha) = a$ and $y'(\alpha) = b$. Thus $f(y) = f(y')$. However, this is a contradiction since $y \in F_a$ and $y' \in F_b$.

COROLLARY 4. *If* X *is normal, then at most* m^p*–many factors* X_α *contain closed discrete subsets of cardinality* m^p.

Remarks. Theorem 3 cannot be strengthened under present hypotheses even if Y is metrizable (a suitable modification of an example in [3, p. 120] will suffice).

Theorems 2 and 3 parts *b*) and *c*) are proved in [1] in the case that p has an immediate cardinal predecessor and Y is metrizable.

If X is equipped with the Tihonov topology, Theorems 1 and 5 and Corollaries 1, 2 and 4 are proved in [5] for separable spaces X_α, and Theorem 2 and Corollaries 1 and 2 are given in [4] for spaces X_α having density m. Also, a variation of Theorem 3 part *b*) is proved in [7].

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