Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the unsteady magnetohydrodynamicflow over yawed infinite cylinder

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **51** (1971), n.5, p. 367–370.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_51_5_367_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1971.

Magnetoidrodinamica. — *On the unsteady magnetohydrodynamic flow over yawed infinite cylinder.* Nota di IOAN POP, presentata (*) dal Socio C. AGOSTINELLI.

RIASSUNTO. — Si studia il moto non stazionario di un fluido conduttore elettrico nella vicinità della linea stagnante di un cilindro infinito che parte impulsivamente dallo stato di riposo e si muove con una velocità costante.

1. The study of a viscous fluid flow past an obstacle has been of great interest for a long time. The flow is supposed to be governed by the Navier-Stokes equations of motion. But the non-linearity of the equations involves many difficulties so that a rigorous mathematical analysis of the general problem is still out of hand. In this short note, the development of unsteady flow of an electrically conducting fluid, impulsively brought in motion with a constant velocity from rest in the vicinity of the rear stagnation line of a yawed infinite cylinder, is considered. This problem has been solved in the non-magnetic case by Roy [1]. Also, recently Katagiri [2, 3] has studied the development of unsteady flow of an electrically conducting fluid started in motion impulsively from rest near the rear stagnation point and at the forward stagnation point of an infinite plane wall which is an electric insulator.

2. The basic equations governing the unsteady motion of incompressible, viscous and electrically conducting fluid in the presence of a constant magnetic field are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\sigma B_0^2}{\rho}u, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) - \frac{\sigma B_0^2}{\rho}v, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

In these equations (x, y, z) are rectangular Cartesian coordinates in which x and y are measured along and respectively parallel to the generators of the body. Then all physical quantities that describe the flow are independent of y. This simplifies the equations and in fact the motion in the xz plane is the same as in two-dimensional flow. Also to simplify the treatement of the

(*) Nella seduta del 13 novembre 1971.

problem, the following assumptions are made as in ref. [2, 3]: (1) The magnetic Reynolds number is very small so that the induced magnetic field can be neglected compared to the applied field. (2) The z component of the magnetic field B_z is constant in space and time, which is denoted by B_0. (3) The electric field is neglected. (4) The usual boundary–layer approximations can be applied.

If for $z \to \infty$, where $\nu = 0$ (ideal fluid), we have $\frac{\partial}{\partial t} = 0$, $\frac{\partial}{\partial z} = 0$ and $w_\infty = \text{const.}$, then the equations (1) to (4) become

$$
(5) \quad \begin{cases} u_\infty \dfrac{\partial u_\infty}{\partial x} = -\dfrac{\sigma B_0^2}{\rho} u_\infty - \dfrac{1}{\rho} \dfrac{\partial p_\infty}{\partial x} \quad \text{or} \quad -\dfrac{1}{\rho} \dfrac{\partial p_\infty}{\partial x} = u_\infty \dfrac{\partial u_\infty}{\partial x} + \dfrac{\sigma B_0^2}{\rho} u_\infty , \\ u_\infty \dfrac{\partial v_\infty}{\partial x} = -\dfrac{\sigma B_0^2}{\rho} v_\infty \quad , \quad \dfrac{\partial p_\infty}{\partial z} = 0 \quad , \quad \dfrac{\partial u_\infty}{\partial x} = 0 . \end{cases}
$$

From (5) we get

$$
(6) \qquad u_\infty = \text{const.} \quad , \quad v_\infty(x) = V \exp\left(-\frac{\sigma B_0^2}{\rho u_\infty} x\right) \quad , \quad p_\infty(x) = p_0 - \sigma B_0^2 \, x u_\infty ,
$$

where V and p_0 are constants.

On the other hand assuming that the velocity of the free stream is as in the non-magnetic case $u_\infty(x) = ax$, then from (5) we find

$$
(7) \quad \begin{cases} -\dfrac{1}{\rho} \dfrac{\partial p_\infty}{\partial x} = a^2 x + \dfrac{\sigma B_0^2}{\rho} ax \quad \text{or} \quad p_\infty(x) = p_0 - \dfrac{1}{2} a (\sigma B_0^2 + \rho a) x^2 , \\ v_\infty(x) = V x^N \qquad \text{where} \quad N = -\dfrac{\sigma B_0^2}{\rho a} , \qquad (N > 0 , a < 0) , \end{cases}
$$

a being a constant and N the parameter of magnetic interaction. This corresponds to a motion in the vicinity of the rear stagnation line of a yawed infinite cylinder.

Now, considering the usual boundary-layer approximations

$$
(8) \quad \begin{cases} u \sim 0(1) \quad , \quad v \sim 0(1) \quad , \quad w \sim 0(\delta) \quad , \quad \nu \sim 0(\delta^2) , \\ \dfrac{\partial u}{\partial t} \sim 0(1) \quad , \quad \dfrac{\partial v}{\partial t} \sim 0(1) \quad , \quad \dfrac{\partial u}{\partial x} \sim 0(1) \quad , \quad \dfrac{\partial v}{\partial x} \sim 0(1) , \\ \dfrac{\partial w}{\partial z} \sim 0(1) \quad , \quad \dfrac{\partial^2 u}{\partial z^2} \sim 0(\delta^{-2}) \quad , \quad \dfrac{\partial^2 v}{\partial z^2} \sim 0(\delta^{-2}) , \end{cases}
$$

where δ is a measure of the thickness of the boundary-layer, the equations (1) to (4) reduce to

$$
(9) \qquad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = u_\infty \frac{\partial u_\infty}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_0^2}{\rho} (u - u_\infty) ,
$$

$$
(10) \qquad \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} = \nu \frac{\partial^2 v}{\partial z^2} - \frac{\sigma B_0^2}{\rho} v ,
$$

$$
(11) \qquad \frac{\partial p}{\partial z} = 0 \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 .
$$

The initial and the boundary conditions are

$$(12)\qquad \begin{aligned} &t \leq 0: && u = v = w = 0 && \text{everywhere,} \\ &t > 0: && u = v = w = 0 && \text{on } z = 0, \\ & && u \to u_\infty(x) \;,\; v \to v_\infty(x) && \text{as } z \to \infty . \end{aligned}$$

3. To solve the equation (9) to (11) it is convenient to write u, v and w as

$$(13)\qquad u = ax \frac{\partial f}{\partial \eta} \;,\quad w = -2a\sqrt{\nu t}\, f \;,\quad v = \mathrm{V} x^{\mathrm{N}} g,$$

where $\eta = z/2\sqrt{\nu t}$ is a similarity variable. We notice that (13) are valid only for small x.

With (13) in mind, the equations (9) and (10) become

$$(14)\qquad \frac{\partial^3 f}{\partial \eta^3} + 2\eta \frac{\partial^2 f}{\partial \eta^2} - 4t \frac{\partial^2 f}{\partial t\, \partial \eta} + 4at \left\{ 1 - \left(\frac{\partial f}{\partial \eta}\right)^2 + f \frac{\partial^2 f}{\partial \eta^2} \right\} - 4at\mathrm{N}\left(1 - \frac{\partial f}{\partial \eta}\right) = 0,$$

$$(15)\qquad \frac{\partial^2 g}{\partial \eta^2} + 2\eta \frac{\partial g}{\partial \eta} - 4t \frac{\partial g}{\partial t} + 4at\left(f \frac{\partial g}{\partial \eta} - \mathrm{N} g \frac{\partial f}{\partial \eta} + \mathrm{N} g\right) = 0,$$

with the boundary conditions

$$(16)\qquad \begin{aligned} &f = \frac{\partial f}{\partial \eta} = g = 0 && \text{on } \eta = 0, \\ &\frac{\partial f}{\partial \eta} \to 1 \;,\; g \to 1 && \text{as } \eta \to \infty . \end{aligned}$$

The equation (14) was solved by Katagiri [2] as it appeared in the two dimensional case. The equation for g is actually solved. We seek the solution of (15) by the form

$$(17)\qquad g(t, \eta) = \sum_{i=0}^{\infty} (at)^i g_i(\eta).$$

Substituting for g into (15) and equating the coefficients of like powers of at we get

$$(18)\qquad \frac{d^2 g_0}{d\eta^2} + 2\eta \frac{dg_0}{d\eta} = 0,$$

$$(19)\qquad \begin{aligned} \frac{d^2 g_i}{d\eta^2} + 2\eta \frac{dg_i}{d\eta} - 4ig_i &= 4\mathrm{N} g_0 \frac{df_0}{d\eta} - 4f_0 \frac{dg_0}{d\eta} - 4\mathrm{N} g_0 \qquad i = 1, \\ &= 4\mathrm{N} \sum_{j=0}^{i-1} \left(g_{i-j-1} \frac{df_j}{d\eta} + g_j \frac{df_{i-j-1}}{d\eta} \right) - \\ &\quad - 4 \sum_{j=0}^{i-1} \left(f_{i-j-1} \frac{dg_i}{d\eta} + f_j \frac{dg_{i-j-1}}{d\eta} \right) - \\ &\quad - 4\mathrm{N} g_{i-1} \qquad i \geq 2, \end{aligned}$$

with

$$(20)\qquad \begin{aligned} & g_0 = 0 \quad , \quad g_i = 0 \qquad \text{on} \quad \eta = 0\,, \\ & g_0 \rightarrow 1 \quad , \quad g_i \rightarrow 0 \qquad \text{as} \quad \eta \rightarrow \infty \quad i \geq 1\,. \end{aligned}$$

The solutions of the zeroth and the first order approximations from (18) and (19) with (20) are

$$(21)\qquad g_0 = \operatorname{erf} \eta\,,$$

$$(22)\qquad g_1 = \frac{2}{\pi}\left(\frac{2}{3} - N\right)(1 + 2\eta^2) + \left\{\frac{1}{2} - N - \frac{2}{\pi}\left(\frac{2}{3} - N\right)\right\} \cdot \\ \cdot \left\{(1 + 2\eta^2)\operatorname{erf}\eta + \frac{2}{\sqrt{\pi}}\,\eta e^{-\eta^2}\right\} - \left\{\frac{1}{2} + (1 - 2N)\,\eta^2\right\}\operatorname{erf}^2\eta + \\ + \left\{\frac{1}{\sqrt{\pi}}(4N - 1)\,\eta e^{-\eta^2} + N\right\}\operatorname{erf}\eta + \frac{2}{\pi}\,N e^{-2\eta^2} - \frac{4}{3\pi}\,e^{-\eta^2},$$

where $\operatorname{erf}\eta$ is the error function

$$(23)\qquad \operatorname{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-s^2}\,ds\,.$$

4. The coefficient of skin-friction, defined as

$$(24)\qquad C_z = \frac{\tau_w}{\rho v_\infty \sqrt{\nu a}} = \frac{1}{2\sqrt{at}}\left(\frac{\partial g}{\partial \eta}\right)_{\eta=0}\,,$$

is calculated in the Table I for various values of N as a function of *at*, where τ_w is the wall shear stress. It shows that the coefficient of skin-friction increases with the magnetic parameter. It may also be noted from Table I that the coefficient of skin-friction decreases in time.

TABLE I.

The coefficient of skin-friction as a function of $2\sqrt{at}$.

$2\sqrt{at}$ \ N	0.0	0.5	1.0	2.0
0.2	5.6503	5.6576	5.6658	5.6808
0.5	2.2779	2.2971	2.3165	2.3551
1.0	1.1709	1.2095	1.2481	1.3248
1.5	0.8162	0.8739	0.9318	1.0474
2.0	0.6494	0.7265	0.8036	0.9577

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