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Union Curvature Tensors and Union Correspondence between two Hypersurfaces of a Finsler Space

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Geometria differenziale. — *Union Curvature Tensors and Union Correspondence between two Hypersurfaces of a Finsler Space.* Nota (*) di MANJULA VERMA, presentata dal Socio E. BOMPIANI.

SUNTO. — Si studiano le proprietà di una corrispondenza puntuale fra due ipersuperficie in uno spazio di Finsler in relazione alle loro « union curves ».

1. INTRODUCTION

Let us consider two hypersurfaces F_n and $\overline{F}_n$ immersed in an $(n+1)$-dimensional Finsler space F_{n+1}. These hypersurfaces are associated with a coordinate system u^α. The coordinates x^i of the space are such that $x = x^i(u^\alpha)$ and $\bar{x}^i = \bar{x}^i(u^\alpha)$ (where, $i = 1, 2, \cdots, n+1$; $\alpha = 1, 2, \cdots, n$) hold for F_n and $\overline{F}_n$ respectively.

Consider a unit vector field λ^i tangential to a congruence of curves in F_{n+1}. At the points of F_n and $\overline{F}_n$ we may write

$$\lambda^i(x, x') = t^{*\alpha}(u, u')\, X^i_\alpha + C^*(u, u')\, n^{*i}(u, u') \tag{1.1a}$$

and

$$\bar{\lambda}^i(x, x') = \bar{t}^{*\alpha}(u, u')\, \overline{X}^i_\alpha + \overline{C}^*(u, u')\, \bar{n}^{*i}(u, u'), \tag{1.1b}$$

where

$$x'^i = \frac{dx^i}{ds}, \quad u'^\alpha = \frac{du^\alpha}{ds}, \quad X^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \quad \overline{X}^i_\alpha = \frac{\partial \bar{x}^i}{\partial u^\alpha}$$

and n^{*i}, $\bar{n}^{*i}$ are the secondary normals to F_n and $\overline{F}_n$ respectively (Rund [2]).

2. UNION CURVES FOR A HYPERSURFACE F_n

The first curvature vectors q^i and p^α of a curve C of F_n are such that

$$q^i = p^\alpha X^i_\alpha + \Omega^*_{\alpha\beta}(u, u') \frac{du^\alpha}{ds} \frac{du^\beta}{ds} n^{*i}, \tag{2.1}$$

where $\Omega^*_{\alpha\beta}$ are the components of the second fundamental tensor of the hypersurface. The differential equations of the union curve of the hypersurface

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are given by

$$(2.2) \qquad p^{\gamma} = \frac{\Omega^{*}_{\alpha\beta} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds}}{C^{*}} \left(t^{*\gamma} - t^{*} \frac{du^{\gamma}}{ds} \right),$$

where

$$t^{*} = g_{\alpha\beta}(u, u') \, t^{*\alpha} \frac{du^{\beta}}{ds},$$

since

$$(2.3) \qquad p^{\gamma} = \frac{d^{2} u^{\gamma}}{ds^{2}} + \Gamma^{*\gamma}_{\beta\tau}(u, u') \frac{du^{\beta}}{ds} \frac{du^{\tau}}{ds},$$

(where $\Gamma^{*\gamma}_{\beta\tau}$ is the induced connection parameter). Therefore union curve of the hypersurface F_n relative to λ^i can be written as

$$(2.4) \qquad \frac{d^{2} u^{\gamma}}{ds^{2}} + U^{*\gamma}_{\beta\tau} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} = 0$$

where

$$(2.5) \qquad U^{*\gamma}_{\beta\tau} = \Gamma^{*\gamma}_{\beta\tau} \frac{K^{*}_{n}}{C^{*}} (\delta^{\gamma}_{\theta} \delta^{\varepsilon}_{\beta} - \delta^{\gamma}_{\beta} \delta^{\varepsilon}_{\theta}) \, t^{*\theta} g_{\tau\varepsilon} .$$

3. Union correspondence between F_n and $\bar{F}_n$.

THEOREM (3.1). *The differential equations of the union curve referred to an arbitrary parameter t are given by*

$$(3.1) \qquad \dot{u}^{\alpha} \ddot{u}^{\beta} - \ddot{u}^{\alpha} \dot{u}^{\beta} = (U^{*\beta}_{\delta\theta} \dot{u}^{\alpha} - U^{*\alpha}_{\delta\theta} \dot{u}^{\beta}) \, \dot{u}^{\delta} \dot{u}^{\theta},$$

where

$$\dot{u}^{\alpha} = \frac{du^{\alpha}}{dt} \quad , \quad \ddot{u}^{\alpha} = \frac{d^{2} u^{\alpha}}{dt^{2}} .$$

Proof. It is easily seen that when we perform parametric transformation $t = t(s)$ $\left(\text{with } \frac{dt}{ds} \neq 0\right)$ equation (2.4) becomes

$$(3.2) \qquad \ddot{u}^{\gamma} + U^{*\gamma}_{\beta\tau} \dot{u}^{\beta} \dot{u}^{\tau} - \dot{u}^{\gamma} \left(\frac{d^{2}s}{dt^{2}} \right) \Big/ \left(\frac{ds}{dt} \right) = 0$$

from which the equation (3.1) follows immediately.

DEFINITION. The hypersurfaces F_n and $\bar{F}_n$ with the metric tensors $g_{\alpha\beta}(x, \dot{x})$, $\bar{g}_{\alpha\beta}(x, \dot{x})$ are said to be in union correspondence (more precisely in u_{λ}-correspondence) if their union curves relative to the congruence λ^i are the same.

Putting

$$\Lambda^{*\alpha}_{\beta\gamma} = \frac{K^*_n}{C^*} (\delta^\alpha_\theta \delta^\varepsilon_\beta - \delta^\alpha_\beta \delta^\varepsilon_\theta) t^{*\theta} g_{\gamma\varepsilon} \tag{3.3}$$

we get after contraction

$$\Lambda^{*\alpha}_{\beta\gamma} = 0 \quad \text{and} \quad \Lambda^{*\alpha}_{\beta\alpha} = (n-1) \frac{K^*_n}{C^*} t^*_\beta . \tag{3.4}$$

In order to study the properties of hypersurfaces in union correspondence we introduce

$$A^\alpha_{\beta\gamma} = \bar{U}^{*\alpha}_{\beta\gamma} - U^{*\alpha}_{\beta\gamma} . \tag{3.5}$$

Defining $\Gamma^*_\beta = \Gamma^{*\alpha}_{\alpha\beta}$ and using the equations (3.4) we get

$$A^\alpha_{\alpha\beta} = \bar{\Gamma}^*_\beta - \Gamma^*_\beta = \varphi_{(1)\beta} \quad \text{(suppose)} \tag{3.6}$$

and

$$A^\alpha_{\beta\alpha} = \varphi_{(1)\beta} + (n-1) \left(\frac{\bar{K}^*_n}{\bar{C}^*} \bar{t}^*_\beta - \frac{K^*_n}{C^*} t^*_\beta \right) = \varphi_{(2)\beta} \quad \text{(suppose)} \tag{3.7}$$

where $g = |g_{\alpha\beta}|$, $\bar{g} = |\bar{g}_{\alpha\beta}|$ and K^*_n, $\bar{K}^*_n$ are normal curvatures to F_n and $\bar{F}_n$ in the direction of a curve C.

THEOREM (3.2). *The necessary and sufficient condition that the two hypersurfaces* F_n *and* $\bar{F}_n$ $(n \neq 3)$, *be in* u_λ*-correspondence is that*

$$\bar{U}^{*\alpha}_{\beta\gamma} = U^{*\alpha}_{\beta\gamma} + \left\{ \frac{1}{(n+1)} \varphi_{(1)\gamma} + \frac{(n-1)^2}{(n+1)(n-3)} \left(\frac{\bar{K}^*_n}{\bar{C}^*} \bar{t}^*_\gamma - \frac{K^*_n}{C^*} t^*_\gamma \right) \right\} \delta^\alpha_\beta + \left\{ \frac{1}{(n+1)} \varphi_{(1)\beta} - \frac{2(n-1)}{(n+1)(n-3)} \left(\frac{\bar{K}^*_n}{\bar{C}^*} \bar{t}^*_\beta - \frac{K^*_n}{C^*} t^*_\beta \right) \right\} \delta^\alpha_\gamma . \tag{3.8}$$

Proof. From (3.1), it is clear that the hypersurfaces are in u_λ-correspondence if and only if

$$(\bar{U}^{*\alpha}_{\delta\theta} - U^{*\alpha}_{\delta\theta}) \dot{u}^\beta \dot{u}^\delta \dot{u}^\theta - (\bar{U}^{*\beta}_{\delta\theta} - U^{*\beta}_{\delta\theta}) \dot{u}^\alpha \dot{u}^\delta \dot{u}^\theta = 0 , \tag{3.9}$$

which in view of (3.5) becomes

$$(A^\alpha_{\delta\theta} \delta^\beta_\gamma - A^\beta_{\delta\theta} \delta^\alpha_\gamma) \dot{u}^\gamma \dot{u}^\delta \dot{u}^\theta = 0 . \tag{3.10}$$

Since this equation holds for arbitrary $\dot{u}^\gamma$, we have

$$A^\alpha_{\delta\theta} \delta^\beta_\gamma + A^\alpha_{\theta\gamma} \delta^\beta_\delta + A^\alpha_{\gamma\delta} \delta^\beta_\theta = A^\beta_{\delta\theta} \delta^\alpha_\gamma + A^\beta_{\theta\gamma} \delta^\alpha_\delta + A^\beta_{\gamma\delta} \delta^\alpha_\theta . \tag{3.11}$$

Contraction of β and γ in (3.11) yields

$$(n-1) A^\alpha_{\delta\theta} + 2 A^\beta_{\theta\delta} = \varphi_{(1)\theta} \delta^\alpha_\delta + \varphi_{(2)\delta} \delta^\alpha_\theta . \tag{3.12}$$

Interchanging δ and θ in (3.12) and solving for $A^{\alpha}_{\delta\theta}$ from the two equations we get

$$(3.13)\quad A^{\alpha}_{\delta\theta} = \frac{1}{(n-3)(n+1)} \{\varphi_{(1)\theta}\delta^{\alpha}_{\delta}(n-1) - 2\varphi_{(2)\theta}\delta^{\alpha}_{\delta} + (n-1)\varphi_{(2)\delta}\delta^{\alpha}_{\theta} - 2\varphi_{(1)\delta}\delta^{\alpha}_{\theta}\}.$$

Simplifying (3.13) with the help of (3.6) and (3.7) we obtain

$$(3.14)\quad A^{\alpha}_{\beta\gamma} = \left\{\frac{1}{(n+1)}\varphi_{(1)\gamma} + \frac{(n-1)^2}{(n+1)(n-3)}\left(\frac{\overline{K}^*_n}{\overline{C}^*}\bar{t}^*_{\gamma} - \frac{K^*_n}{C^*}t^*_{\gamma}\right)\right\}\delta^{\alpha}_{\beta} + \left\{\frac{1}{(n+1)}\varphi_{(1)\beta} - \frac{2(n-1)}{(n+1)(n-3)}\left(\frac{\overline{K}^*_n}{\overline{C}^*}\bar{t}^*_{\beta} - \frac{K^*_n}{C^*}t^*_{\beta}\right)\right\}\delta^{\alpha}_{\gamma}.$$

Substituting in (3.5) the value of $A^{\alpha}_{\beta\gamma}$ from (3.14) we deduce the required equation (3.8) as required.

For $n = 3$, the equation (3.12) determines only the symmetric part of $A^{\alpha}_{\beta\gamma}$. However, we shall assume that $n \neq 3$.

Suppose, in particular, that the congruence λ^i is normal to the subspace $\overline{F}_n$. We have then $\Lambda^{*\alpha}_{\beta\gamma} = 0$ and the union curve of $\overline{F}_n$ is a geodesic of the hypersurface (Mishra and Singh [1]). The Theorem (3.2) will, therefore, take the form,

THEOREM (3.3). *A union curve of the hypersurface* F_n *will correspond to a geodesic of* $\overline{F}_n$ *if*

$$(3.15)\quad \overline{\Gamma}^{*\alpha}_{\beta\gamma} = U^{*\alpha}_{\beta\gamma} + \left\{\frac{1}{(n+1)}\varphi_{(1)\gamma} - \frac{(n-1)^2}{(n+1)(n-3)}\frac{K^*_n}{C^*}t^*_{\gamma}\right\}\delta^{\alpha}_{\beta} + \left\{\frac{1}{(n+1)}\varphi_{(2)\beta} + \frac{2(n-1)}{(n+1)(n-3)}\frac{K^*_n}{C^*}t^*_{\beta}\right\}\delta^{\alpha}_{\gamma}.$$

Proceeding as in the case of Theorem (3.2), we can deduce that in order that the spaces F_n and $\overline{F}_n$ may be in geodesic correspondence (that is, a geodesic of F_n may correspond to geodesic of $\overline{F}_n$) it is necessary and sufficient that

$$(3.16)\quad \overline{\Gamma}^{*\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} + \frac{1}{(n+1)}\{\varphi_{(1)\gamma}\delta^{\alpha}_{\beta} + \varphi_{(1)\beta}\delta^{\alpha}_{\gamma}\}.$$

Putting

$$U^{\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} + \Lambda^{*\alpha}_{\beta\gamma}$$

and corresponding equation for $\overline{F}_n$ in (3.8) we get

$$(3.17)\quad \overline{\Gamma}^{*\alpha}_{\beta\gamma} - \Gamma^{*\alpha}_{\beta\gamma} - \frac{1}{(n+1)}(\varphi_{(1)\gamma}\delta^{\alpha}_{\beta} + \varphi_{(1)\beta}\delta^{\alpha}_{\gamma}) = \Lambda^{*\alpha}_{\beta\gamma} - \overline{\Lambda}^{*\alpha}_{\beta\gamma} + \frac{(n-1)}{(n+1)(n-3)}\left\{(n-1)\left(\frac{\overline{K}^*_n}{\overline{C}^*}\bar{t}^*_{\gamma} - \frac{K^*_n}{C^*}t^*_{\gamma}\right)\delta^{\alpha}_{\beta} - 2\left(\frac{\overline{K}^*_n}{\overline{C}^*}\bar{t}^*_{\beta} - \frac{K^*_n}{C^*}t^*_{\beta}\right)\delta^{\alpha}_{\gamma}\right\}.$$

THEOREM (3.4) *a. A necessary and sufficient condition that the u_λ–correspondence between the hypersurfaces* F_n *and* $\overline{F}_n$ *may yield the geodesic correspondence between the spaces is that the tensor*

$$B^{\alpha}_{\beta\gamma} = \Lambda^{*\alpha}_{\beta\gamma} - \frac{(n-1)^2}{(n+1)(n-3)} \frac{K^*_n}{C^*} t^*_\gamma \delta^\alpha_\beta + \frac{2(n-1)}{(n+1)(n-3)} \frac{K^*_n}{C^*} t^*_\beta \delta^\alpha_\gamma ,$$

is invariant relative to u_λ–correspondence.

The following Theorem is an immediate consequence of equation (3.8).

THEOREM (3.4) *b. If the hypersurfaces* F_n *and* $\overline{F}_n$ *are in u_λ–correspondence then the connection parameter*

$$V^{*\alpha}_{\beta\gamma} = U^{*\alpha}_{\beta\gamma} - \frac{1}{(n+1)} \left[\left(\Gamma^*_\gamma + \frac{(n-1)^2}{(n-3)} \frac{K^*_n}{C^*} t^*_\gamma\right)\delta^\alpha_\beta - \left(\Gamma^*_\beta + \frac{2(n-1)}{(n-3)} \frac{K^*_n}{C^*} t^*_\beta\right)\delta^\alpha_\gamma\right]$$

is invariant.

4. UNION CURVATURE TENSOR

It is obvious from (2.5) that $U^{*\alpha}_{\beta\gamma}$ is an asymmetric connection, called union connection parameter. The connection $U^{*\alpha}_{\beta\gamma}$ will be used in defining the following two methods of covariant differentiation of a tensor of the type $T^\alpha_\beta(u, u')$ (Verma [3]).

(4.1) $$T^\alpha_{\beta,\gamma} = \frac{\partial T^\alpha_\beta}{\partial u^\gamma} + \frac{\partial T^\alpha_\beta}{\partial u'^\delta} \frac{\partial u'^\delta}{\partial u^\gamma} + T^\rho_\beta U^{*\alpha}_{\rho\gamma} - T^\alpha_\rho U^{*\rho}_{\beta\gamma}$$

and

(4.2) $$T^\alpha_{\beta;\gamma} = \frac{\partial T^\alpha_\beta}{\partial u^\gamma} + \frac{\partial T^\alpha_\beta}{\partial u'^\delta} \frac{\partial u'^\delta}{\partial u^\gamma} + T^\rho_\beta U^{*\alpha}_{\gamma\rho} - T^\alpha_\rho U^{*\alpha}_{\gamma\beta} .$$

Suppose ϑ^α are the contravariant components of a vector field of the hypersurface. The above two methods of covariant differentiation yield the commulative formulae

(4.3) $$\vartheta^\alpha_{,\beta\gamma} - \vartheta^\alpha_{,\gamma\beta} = \vartheta^\delta \underset{1}{R}{}^\alpha_{.\delta\gamma\beta} + \vartheta^\alpha_{,\theta}(U^{*\theta}_{\gamma\beta} - U^{*\theta}_{\beta\gamma})$$

and

(4.4) $$\vartheta^\alpha_{;\beta\gamma} - \vartheta^\alpha_{;\gamma\beta} = \vartheta^\delta \underset{2}{R}{}_{.\delta\gamma\beta} + \vartheta^\alpha_{;\theta}(U^{*\theta}_{\beta\gamma} - U^{*\theta}_{\gamma\beta})$$

where

(4.5) $$\underset{1}{R}{}^\sigma_{\tau\alpha\beta} = \left(\frac{\partial U^{*\sigma}_{\tau\alpha}}{\partial u^\beta} + \frac{\partial U^{*\sigma}_{\tau\alpha}}{\partial u'^\delta} \frac{\partial u'^\delta}{\partial u^\beta}\right) -$$
$$- \left(\frac{\partial U^{*\sigma}_{\tau\beta}}{\partial u^\alpha} + \frac{\partial U^{*\sigma}_{\tau\beta}}{\partial u'^\delta} \frac{\partial u'^\delta}{\partial u^\alpha}\right) + U^{*\sigma}_{\gamma\beta} U^{*\gamma}_{\tau\alpha} - U^{*\sigma}_{\gamma\alpha} U^{*\gamma}_{\tau\beta}$$

and

$$\underset{2}{R}{}^{\sigma}{}_{\tau\alpha\beta} = \left(\frac{\partial U^{*\sigma}_{\alpha\tau}}{\partial u^{\beta}} + \frac{\partial U^{*\sigma}_{\alpha\tau}}{\partial u'^{\delta}} \frac{\partial u'^{\delta}}{\partial u^{\beta}}\right) -$$

$$- \left(\frac{\partial U^{*\sigma}_{\beta\tau}}{\partial u^{\alpha}} + \frac{\partial U^{*\sigma}_{\beta\tau}}{\partial u'^{\delta}} \frac{\partial u'^{\delta}}{\partial u^{\alpha}}\right) + U^{*\alpha}_{\beta\gamma} U^{*\gamma}_{\alpha\tau} - U^{*\sigma}_{\alpha\gamma} U^{*\gamma}_{\beta\tau}$$

are called the union curvature tensors of the first and second kinds of the hypersurface.

Let us suppose that

$$\mu_{\beta} = \frac{1}{(n+1)} \varphi_{(1)\beta} + \frac{(n-1)^2}{(n+1)(n-3)} \left(\frac{\bar{K}^*_n}{\bar{C}^*} \bar{t}^*_{\beta} - \frac{K^*_n}{C^*} t^*_{\beta}\right) \tag{4.7)a}$$

and

$$\nu_{\tau} = \frac{1}{(n+1)} \varphi_{(1)\tau} - \frac{2(n-1)}{(n+1)(n-3)} \left(\frac{\bar{K}^*_n}{\bar{C}^*} \bar{t}^*_{\tau} - \frac{K^*_n}{C^*} t^*_{\tau}\right). \tag{4.7)b}$$

Thus the equation (3.8) can be written in the following form —

$$\bar{U}^{*\sigma}_{\tau\beta} = U^{*\sigma}_{\tau\beta} + \mu_{\beta} \delta^{\sigma}_{\tau} + \nu_{\tau} \delta^{\sigma}_{\beta} . \tag{4.8}$$

Using the expression for $\underset{1}{R}{}^{\sigma}{}_{\tau\alpha\beta}$ and substituting the value of $\bar{U}^{*\sigma}_{\tau\beta}$ from (4.8) in the corresponding expression for $\underset{1}{\bar{R}}{}^{\sigma}{}_{\tau\alpha\beta}$ (the union curvature tensor of the first kind of $\bar{F}_n$) and noting the fact that

$$U^{*\sigma}_{\tau\beta} - U^{*\sigma}_{\beta\tau} = \frac{K^*_n}{C^*} (\delta^{\sigma}_{\beta} t^*_{\tau} - \delta^{\sigma}_{\tau} t^*_{\beta}) \tag{4.9}$$

we obtain after some simplifications

$$\underset{1}{\bar{R}}{}^{\sigma}{}_{\tau\alpha\sigma} = \underset{1}{R}{}^{\sigma}{}_{\tau\alpha\sigma} + \delta^{\sigma}_{\tau}(\underset{1}{M}_{\beta\alpha} - \underset{1}{M}_{\alpha\beta}) - \delta^{\sigma}_{\alpha} \underset{1}{N}_{\tau\beta} + \delta^{\sigma}_{\beta} \underset{1}{N}_{\tau\alpha} , \tag{4.10}$$

where

$$\underset{1}{M}_{\alpha\beta} = \mu_{\alpha,\beta} - \mu_{\alpha} \mu_{\beta} - \frac{K^*_n}{C^*} \mu_{\alpha} t^*_{\beta} \tag{4.10)a}$$

and

$$\underset{1}{N}_{\tau\beta} = \nu_{\tau,\beta} - \nu_{\tau} \nu_{\beta} - \frac{K^*_n}{C^*} \nu_{\tau} t^*_{\beta} \tag{4.10)b}$$

and where we have also used the fact

$$U^{*\alpha}_{\beta\gamma} - U^{*\alpha}_{\gamma\beta} = \Lambda^{*\alpha}_{\beta\gamma} - \Lambda^{*\alpha}_{\gamma\beta} = \frac{K^*_n}{C^*} (\delta^{\alpha}_{\gamma} t^*_{\beta} - \delta^{\alpha}_{\beta} t^*_{\gamma}) .$$

Further we have

$$\underset{1}{\bar{R}}{}^{\sigma}{}_{\tau\alpha\sigma} = \underset{1}{\bar{R}}_{\tau\alpha} = \underset{1}{R}_{\tau\alpha} + (\underset{1}{M}_{\tau\alpha} - \underset{1}{M}_{\alpha\tau}) + (n-1) \underset{1}{N}_{\tau\alpha} . \tag{4.11}$$

THEOREM (4.1). *The union curvature tensors of the first kind for the spaces* $\overline{F}_n$, F_n *are given by* (4.10) *and the union Ricci tensors for the two spaces satisfy* (4.4).

A similar calculation based on the equations (4.6) and (4.8) yields the following Theorem.

THEOREM (4.2). *The union curvature tensors of the second kind for the spaces* $\overline{F}_n$, F_n *are given by*

$$\underset{2}{\overline{R}}{}^{\sigma}{}_{\tau\alpha\sigma} = \underset{2}{R}{}^{\sigma}{}_{\tau\alpha\sigma} + \delta^{\sigma}_{\tau}(\underset{2}{N}_{\beta\alpha} - \underset{2}{N}_{\alpha\beta}) - \delta^{\sigma}_{\alpha}\underset{2}{M}_{\tau\beta} + \delta^{\sigma}_{\beta}\underset{2}{M}_{\tau\alpha}. \tag{4.12}$$

where

$$\underset{2}{M}_{\alpha\beta} = \mu_{\alpha;\beta} - \mu_{\alpha}\mu_{\beta} + \frac{K^*_n}{C^*}\mu_{\alpha} t^*_{\beta} \tag{4.13a}$$

and

$$\underset{2}{N}_{\tau\beta} = \nu_{\tau;\beta} - \nu_{\tau}\nu_{\beta} + \frac{K^*_n}{C^*}\nu_{\tau} t^*_{\beta}, \tag{4.13b}$$

and the union Ricci tensors of the second kind for the two spaces satisfy

$$\underset{2}{\overline{R}}_{\tau\alpha} = \underset{2}{\overline{R}}{}^{\sigma}{}_{\tau\alpha\sigma} = \underset{2}{R}_{\tau\alpha} + (\underset{1}{N}_{\tau\alpha} - \underset{1}{N}_{\alpha\tau}) + (n-1)\underset{1}{M}_{\tau\alpha}. \tag{4.13}$$

The proof of this Theorem is similar to that of the Theorem (4.2).

5. INVARIANTS ARISING OUT OF u_λ–CORRESPONDENCE

In order to obtain an invariant involving $\underset{1}{R}{}^{\sigma}{}_{\tau\alpha\beta}$ and $\underset{2}{R}{}^{\sigma}{}_{\tau\alpha\beta}$ several relations which are given below will be used.

Let us suppose that

$$\lambda_{(1)\beta} = \frac{1}{(n+1)}\Gamma^*_{\beta} + \frac{(n-1)^2}{(n+1)(n-3)}\frac{K^*_n}{C^*}t^*_{\beta} \tag{5.1a}$$

and

$$\overline{\lambda}_{(1)\beta} = \frac{1}{(n+1)}\overline{\Gamma}^*_{\beta} + \frac{(n-1)^2}{(n+1)(n-3)}\frac{\overline{K}^*_n}{\overline{C}^*}\overline{t}^*_{\beta}. \tag{5.1b}$$

Subtracting (5.1)a from (5.1)b and making use of the equations (3.6) and (4.7)a we get

$$\mu_{\beta} = \overline{\lambda}_{(1)\beta} - \lambda_{(1)\beta}. \tag{5.2}$$

Further suppose that

$$\lambda_{(2)\beta} = \frac{1}{(n+1)}\Gamma^*_{\beta} - \frac{2(n-1)}{(n+1)(n-3)}\frac{K^*_n}{C^*}t^*_{\beta} \tag{5.3a}$$

and

(5.3)b $$\bar{\lambda}_{(2)\beta} = \frac{1}{(n+1)} \bar{\Gamma}^*_\beta - \frac{2(n-1)}{(n+1)(n-3)} \frac{\bar{K}^*_n}{\bar{C}^*} \bar{t}^*_\beta .$$

Subtracting (5.3)a from (5.3)b and using the equations (3.6) and (4.7)b we obtain

(5.4) $$\nu_\beta = \bar{\lambda}_{(2)\beta} - \lambda_{(2)\beta} .$$

Using the equations of the type (2.5) we get (after some simplifications)

(5.5) $$\mu_\gamma (U^\gamma_{\alpha\beta} - U^\gamma_{\beta\alpha}) = \frac{K^*_n}{C^*} (\mu_\beta t^*_\alpha - \mu_\alpha t^*_\beta)$$

and

(5.6) $$\frac{K^*_n}{C^*} t^*_\beta = \frac{(n-3)}{(n-1)} (\lambda_{(1)\beta} - \lambda_{(2)\beta}) .$$

Using the equation (4.8) and denoting by $\bar{\lambda}_{(1)\alpha\bar{;}\beta}$ and $\bar{\lambda}_{(2)\alpha\bar{;}\beta}$ the covariant differentiations of the type (4.1) with respect to the connection parameter $\bar{U}^{*\alpha}_{\beta\gamma}$, we get after some simplifications

(5.7)a $$\bar{\lambda}_{(1)\alpha\bar{;}\beta} = \bar{\lambda}_{(1)\alpha,\beta} - \mu_\beta \bar{\lambda}_{(1)\alpha} + \nu_\alpha \bar{\lambda}_{(1)\beta}$$

and

(5.7)b $$\bar{\lambda}_{(2)\tau\bar{;}\alpha} = \bar{\lambda}_{(2)\tau\bar{;}\alpha} - \mu_\alpha \bar{\lambda}_{(2)\tau} + \nu_\tau \bar{\lambda}_{(2)\alpha} .$$

THEOREM (5.1). *The tensor* $\underset{1}{S}{}^\sigma{}_{\tau\alpha\beta}$ *defined by*

(5.8) $$\underset{1}{S}{}^\sigma{}_{\tau\alpha\beta} = \underset{1}{R}{}^\sigma{}_{\tau\alpha\beta} + \left\{ \lambda_{(1)\alpha,\beta} - \lambda_{(1)\beta,\alpha} + \frac{(n-3)}{(n-1)} (\lambda_{(1)\alpha} \lambda_{(2)\beta} - \lambda_{(1)\beta} \lambda_{(2)\alpha}) \right\} \delta^\sigma_\tau - \\ - \left\{ \lambda_{(2)\tau,\alpha} + \frac{2(n-2)}{(n-1)} \lambda_{(2)\tau} \lambda_{(2)\alpha} - \frac{(n-3)}{(n-1)} \lambda_{(2)\tau} \lambda_{(1)\alpha} \right\} \delta^\sigma_\beta + \\ + \left\{ \lambda_{(2)\tau,\beta} + \frac{2(n-2)}{(n-1)} \lambda_{(2)\tau} \lambda_{(2)\beta} - \frac{(n-3)}{(n-1)} \lambda_{(2)\tau} \lambda_{(1)\beta} \right\} \delta^\sigma_\alpha ,$$

is an invariant with respect to u_λ*-correspondence.*

Proof. Using the equation (5.5), (5.7)a and (5.7)b and (4.8) we deduce

(5.9) $$\underset{1}{M}_{\alpha\beta} - \underset{1}{M}_{\beta\alpha} = (\bar{\lambda}_{(1)\alpha\bar{;}\beta} - \bar{\lambda}_{(1)\beta\bar{;}\alpha}) - (\lambda_{(1)\alpha,\beta} - \lambda_{(1)\beta,\alpha}) + \\ + \bar{\lambda}_{(1)\gamma} (\bar{U}^{*\gamma}_{\beta\alpha} - \bar{U}^{*\gamma}_{\alpha\beta}) - \lambda_{(1)\gamma} (U^{*\gamma}_{\beta\alpha} - U^{*\gamma}_{\alpha\beta})$$

and

(5.10) $$\underset{1}{N}_{\tau\alpha} \delta^\sigma_\beta - \underset{1}{N}_{\tau\beta} \delta^\sigma_\alpha = \delta^\sigma_\beta (\bar{\lambda}_{(2)\tau\bar{;}\alpha} - \lambda_{(2)\tau\bar{;}\alpha}) + (\bar{\lambda}_{(2)\tau} \bar{\lambda}_{(2)\alpha} - \lambda_{(2)\tau} \lambda_{(2)\alpha}) - \\ - (\bar{\lambda}_{(2)\tau\bar{;}\alpha} - \lambda_{(2)\tau,\alpha} + \bar{\lambda}_{(2)\tau} \bar{\lambda}_{(2)\alpha} - \lambda_{(2)\tau} \lambda_{(2)\alpha}) - \bar{\lambda}_{(2)\tau} (\bar{U}^{*\sigma}_{\alpha\beta} - \bar{U}^{*\sigma}_{\beta\alpha}) + \\ + \lambda_{(2)\tau} (U^{*\sigma}_{\alpha\beta} - U^{*\sigma}_{\beta\alpha}) .$$

Substituting the expressions (5.9) and (5.10) in (4.10) and using (4.9), we get

$$\overline{\underset{1}{S}}{}^{\sigma}{}_{\tau\alpha\beta} = \underset{1}{S}{}^{\sigma}{}_{\tau\alpha\beta} ,$$

which proves the Theorem.

THEOREM (5.2). *The tensor* $\underset{1}{S}_{\tau\alpha}$ *defined by*

$$(5.11) \qquad \underset{1}{S}_{\tau\alpha} = \underset{1}{\overline{R}}_{\tau\alpha} + \lambda_{(1)\alpha,\tau} - \lambda_{(1)\tau,\alpha} + \frac{n(n-3)}{(n-1)} \lambda_{(1)\alpha}\lambda_{(2)\tau} - \frac{(n-3)}{(n-1)} \lambda_{(1)\tau}\lambda_{(2)\alpha} - (n-1)\lambda_{(2)\tau,\alpha} - 2(n-2)\lambda_{(2)\tau}\lambda_{(2)\alpha} ,$$

is an invariant with respect to u_λ*–correspondence.*

Proof. In the above $\underset{1}{R}_{\tau\alpha} = \underset{1}{R}{}^{\sigma}{}_{\tau\alpha\sigma}$ is the union Ricci tensor of the first kind. Contracting σ and β in (5.8) we get the equation (5.11).

A similar calculation based on the equations (5.2), (5.4), (5.5), (5.6), (4.8), (5.7)a and (5.7)b yields the following Theorems.

THEOREM (5.3). *The tensors* $\underset{2}{S}{}^{\sigma}{}_{\tau\alpha\beta}$ *and* $\underset{2}{S}_{\tau\alpha}$ *defined by*

$$(5.12) \qquad \underset{2}{S}{}^{\sigma}{}_{\tau\alpha\beta} = \underset{2}{R}{}^{\sigma}{}_{\tau\alpha\beta} + \delta^{\sigma}_{\tau}\left\{\lambda_{(2)\alpha;\beta} - \lambda_{(2)\beta;\alpha} - \frac{(n-3)}{(n-1)}(\lambda_{(1)\alpha}\lambda_{(2)\beta} - \lambda_{(1)\beta}\lambda_{(2)\alpha})\right\} + \delta^{\sigma}_{\alpha}\left\{\lambda_{(1)\tau;\beta} + \frac{2(n-2)}{(n-1)}\lambda_{(1)\tau}\lambda_{(1)\beta} - \frac{(n-3)}{(n-1)}\lambda_{(1)\tau}\lambda_{(2)\beta}\right\} - \delta^{\sigma}_{\beta}\left\{\lambda_{(1)\tau;\alpha} + \frac{2(n-2)}{(n-1)}\lambda_{(1)\tau}\lambda_{(1)\alpha} - \frac{(n-3)}{(n-1)}\lambda_{(1)\tau}\lambda_{(2)\alpha}\right\}$$

and

$$(5.13) \qquad \underset{2}{S}_{\tau\alpha} = \underset{2}{R}_{\tau\alpha} + \lambda_{(2)\alpha;\tau} - \lambda_{(2)\tau;\alpha} - \frac{(n-3)}{(n-1)}\lambda_{(2)\tau}\lambda_{(1)\alpha} + \frac{n(n-3)}{(n-1)}\lambda_{(1)\tau}\lambda_{(2)\alpha} - (n-1)\lambda_{(1)\tau;\alpha} - 2(n-2)\lambda_{(1)\tau}\lambda_{(1)\alpha} .$$

are invariant with respect to u_λ*–correspondence.*

$\underset{2}{R}_{\tau\alpha} = \underset{2}{R}{}^{\sigma}{}_{\cdot\tau\alpha\sigma}$ is the union Ricci tensor of the second kind in the above Theorem.

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