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Geometrie regularity and formal smoothness

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Geometria. — *Geometric regularity and formal smoothness.* Nota di A. BREZULEANU e N. RADU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota si generalizza il criterio 22.5.8, *EGA* dato da A. Grothendieck (ved. la Bibliografia).

For the notations and terminology see *EGA*, § 19.

We shall consider a local morphism of local rings $A \to B$; A and B have the topologies given by their maximal ideals.

DEFINITION. B is geometrically regular over A if, for any localisation A' of a finite A–algebra in a maximal ideal, such that A' is a regular ring, $B \otimes_A A'$ is also a regular ring.

THEOREM. *Let B be noetherian.*

(i) *If B is a formally smooth A–algebra, then B is geometrically regular over A.*

(ii) *Assume that it exists an injective, finite morphism of rings $A \to A''$ such that A'' is a regular (noetherian) ring. If B is geometrically regular over A, then B is a formally smooth A–algebra.*

Proof. The first part is an easy consequence of the following known fact: let A and B be noetherian rings and B a formally smooth A–algebra, then A is regular if and only if B is regular.

If there is an injective and finite morphism $A \to A''$, with A'' noetherian, it follows that A is also noetherian [3]. Let B be geometrically regular over A. Let k be the residue class field of A. By definition, $B \otimes_A k$ is geometrically regular over k, i.e. formally smooth over k (*EGA*, 22.5.8). Then the second statement of the Theorem is proved, if B is a flat A–module (*EGA*, 19.7.1).

Using [4], it suffices to prove that $A'' \otimes_A B$ is a flat A''–module.

Since any localisation B' of $A'' \otimes_A B$ in a maximal ideal is geometrically regular over a corresponding localisation A' of A'' in a maximal ideal, it suffices to show that such B' are flat A'–modules. Let $x_1, \cdots, x_n$ be a regular system of parameters of A'. Then the rings $B' \Big/ \sum_{j=1}^{i} x_j B'$, $i \leq n$ are regular, since B' is geometrically regular over A'. If follows that the sequence $x_1, \cdots, x_n$ is B'–regular, hence B' is a flat A'–module (*EGA*, 15.1.21).

COROLLARY. *Let $A \to B$ be a local morphism of local noetherian rings, with A a japanese domain of dimension 1. Then B is geometrically regular over A if and only if B is a formally smooth A–algebra.*

(*) Nella seduta del 13 novembre 1971.

Remarks: (*i*) Let $u : A \to B$ be a surjective local morphism of artinian rings. Then B is geometrically regular over A; but B is a formally smooth A–algebra only if u is an isomorphism (since, in this case, u would be flat).

(ii) If A is a field then the geometric regularity localises. If A is a regular ring and $\dim A \geq 1$, this is not true in general, a counterexample being $E3$, pp. 207 in *LR*. By the above result, if the formal smoothness of $A \to B$ localizes, then the geometric regularity of $A \to B$ localizes also.

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