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Algebra topologica. — *Function Algebras over Valued Fields and Measures.* Nota I di GEORGE BACHMAN, EDWARD BECKENSTEIN e LAWRENCE NARICI (*), presentata (**), dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si studia l'algebra topologica F(T) delle funzioni continue che applicano uno spazio O–dimensionale T in un campo valutato non archimedeo completo, munito della topologia compatto–aperta.

In this paper we study the structure of the topological algebra F(T) of continuous functions taking a O–dimensional space T into a complete nonarchimedean valued field F endowed with compact-open topology. In particular we completely characterize the dual space F(T)′ of F(T). In [2] we showed that a semisimple complete barreled Q–algebra X is a uniform algebra if and only if the spectral radius norm generates a topology of the dual pair (X , X′). In this paper (Theorem 1) we develop a version of this result for topological algebras over nonarchimedean valued fields.

In [3] we developed a notion of support of a continuous linear functional on F(T). There we showed that when T is a Lindelöf space, the support has properties which make it useful in a number of applications. The development of the support's properties was effected without reference to a measure on T or representation theory. The support notion was then used to prove an analog of a classical Theorem of Nachbin [9, p. 471] in which a necessary and sufficient condition for F(T) to be F–barreled when F(T) carries the compact-open topology was developed.

Here we develop a Riesz-type representation theory for F(T) and show that the properties of support developed in [3] for Lindelöf spaces T also obtain if T is merely O–dimensional (Theorem 2).

In the last section, Sec. 3, of this paper we consider zero-one measures on T taking values in a nonarchimedean valued field F where F is assumed not to contain $\sqrt{-1}$. We show that the homomorphisms of F(T) into F are just the evaluation maps if and only if a certain collection of zero-one measures have nonempty support. We then show that if F is a local field without $\sqrt{-1}$ and T is a Lindelöf space, then the homomorphisms of F(T) into F are just the evaluation maps.

In Sec. 3 we also construct a zero-one measure with empty support (Example 2). From this example we show that there are bounded measures with compact support on T which do not generate continuous linear functionals on F(T). We also exhibit a discontinuous linear functional on a dense subalgebra of F(T).

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1. Representation theory when T is compact

Throughout this section T is a O–dimensional compact Hausdorff space, F a complete nontrivially valued nonarchimedean field, and F(T) the topological algebra of continuous functions (with pointwise operations) mapping T into F with the sup-norm topology. The collection of clopen (= closed and open) subsets of T is denoted by $\mathcal{S}$.

Definition 1. A *bounded measure* on T is a function μ mapping $\mathcal{S}$ into F satisfying the following two conditions:

(*a*) for $U, V \in \mathcal{S}$ such that $U \cap V = \varnothing$, $\mu(U \cup V) = \mu(U) + \mu(V)$;

(*b*) there exists $N > 0$ such that for each $U \in \mathcal{S}$, $|\mu(U)| \leq N$.

Van Rooij and Schikof ([13]) considered measures on O–dimensional spaces and defined the notion of a measure in a slightly different manner. In the case when T is compact, however, Definition 1 yields the same type of measure developed in [13].

When one has a measure on T satisfying the conditions of Definition 1 a "Riemann integral" of function f in F(T) may be defined as follows: Let $P = \{U_1, \cdots, U_n\}$ be a family of clopen sets which form a disjoint partition of T, let x_i be any point in U_i, let $V_i(f)$ denote the oscillation of f on U_i (which must be finite for each i, $1 \leq i \leq n$, and let $d_P = \sup_i V_i(f)$. Then

$$\int_T f \, d\mu = \lim_{d_P \to 0} \sum_{i=1}^{n} f(x_i) \, \mu(U_i).$$

As is shown in [8], $\int_T f \, d\mu$ exists for all $f \in F(T)$; other properties of $\int_T \cdot \, d\mu$ are also discussed there. At times $\int_T f \, d\mu$ will be denoted as $I(f)$.

This integral is of course a linear functional mapping F(T) into F. If $U \in \mathcal{S}$ and k_U is the characteristic function of U, then $I(k_U) = \mu(U)$. Moreover $|I(f)| \leq N \|f\|$ where N is as in Definition 1, and $\|f\| = \sup |f(T)|$. Thus I is a continuous linear functional on F(T). Conversely we have:

Proposition 1. *Let* $J \in F(T)'$. *Then there is a bounded measure* μ *on* T *such that* $J(f) = \int_T f \, d\mu$.

Proof. For each $U \in \mathcal{S}$ take $\mu(U)$ to be $J(k_U)$. Since J is continuous, there exists $N > 0$ such that $|J(k_U)| \leq N \|k_U\| = N$. Thus μ is a bounded measure.

Since for any $U_1, \cdots, U_n \in \mathcal{S}$, and $x_i \in U_i$ for each i,

$$J\left(\sum_{i=1}^{n} f(x_i) \, k_{U_i}\right) = \sum_{i=1}^{n} f(x_i) \, J(k_{U_i}) = \sum_{i=1}^{n} f(x_i) \, \mu(U_i)$$

then, by the continuity of J and the fact that $\Sigma f(x_i)\, k_{U_i}$ converges uniformly to f as $\sup_i V_i(f) \to 0$, $J(f) = \int_T f\, d\mu$.

Clearly the functions of the form $\Sigma f(x_i)\, k_{U_i}$ are dense in F(T), so that the map $J \to \mu$ introduced in Proposition 1 is injective.

Let B(T) denote the set of bounded measures on T. With respect to the natural operations of addition of bounded measures and multiplication by a scalar, B(T) is a vector space over F. It is a normed space with respect to the norm introduced in the definition below.

DEFINITION 2. For $\mu \in B(T)$, the *norm* of μ, is given by $\|\mu\| = \sup_{U \in \mathcal{S}} |\mu(U)|$.

PROPOSITION 2. *The continuous dual* F(T)′ *of* F(T) *is isometrically isomorphic to* B(T).

Proof. Let $J \in F(T)'$. We map J into the following $\mu \in B(T)$: If $U \in \mathcal{S}$, take $\mu(U) = J(k_U)$. It is evident that the map $J \to \mu$ of Proposition 1 establishes an algebraic isomorphism between F(T)′ and B(T). To show that it is also an isometry, it only remains to show that $\|J\| = \sup_{f \neq 0} |J(f)|/\|f\| = \sup_{U \in \mathcal{S}} |\mu(U)|$. Since $|J(f)| \leq \|f\|\, \|\mu\|$ for any $f \in F(T)$ by previous arguments, it is clear that $\|J\| \leq \|\mu\|$. To obtain the reverse inequality, let $\varepsilon > 0$ be given and choose $U \in \mathcal{S}$ such that $|\mu(U)| > \|\mu\| - \varepsilon$. Hence $|J(k_U)| = |\mu(U)| = |J(k_U)|/\|k_U\|$ implies $\|J\| > \|\mu\| - \varepsilon$ for all $\varepsilon > 0$. This completes the proof.

As a result of this representation Theorem for the dual space of the V*-algebra F(T) [10, p. 148] and the applicability of the Hahn-Banach Theorem for normed linear spaces over spherically complete fields, we may now state an analog of [2, Theorem 4]:

THEOREM 1. *Let* X *be a commutative complete nonarchimedean locally multiplicatively* F*–convex algebra with identity over a local field* F. *Assume that* X *is a semisimple* F*–barreled* Q*–algebra and a projective limit of a family of Gelfand algebras* [10, p. 106]. *Then* X *is a uniform algebra if and only if the topology generated by the spectral radius norm is a topology of the dual pair* (X , X′).

2. REPRESENTATION THEORY WHEN T IS NOT COMPACT

In this section T is a O–dimensional Hausdorff space and F is a discretely valued field. F(T) carries the compact-open topology.

DEFINITION 1. Let μ be a bounded measure on T and let $\mathfrak{D}_\mu$ denote the set of all subsets $U \in \mathcal{S}$ such that for each clopen subset W of U, $\mu(W) = 0$.

DEFINITION 2. The *support* F_μ of a bounded measure μ is the complement $C(\cup \mathfrak{D}_\mu)$ of $\cup \mathfrak{D}_\mu$. If F_μ is compact, then μ is said to have *compact support.*

We show in the next section that there is a bounded measure with compact support which is not a measure in the sense of van Rooij and Schikof ([13]). In this section we obtain a $1-1$ correspondence between certain bounded measures with compact support and continuous linear functionals on F(T). We also show in the next section that there is a bounded measure with compact support which does not produce a continuous linear functional.

DEFINITION 3. Let L be a compact subset of T. L is called a *bounding set* for the linear functional $J \in F(T)'$ if there exists $N_L > 0$ such that $|J(f)| \leq N_L \sup_{t \in L} |f(t)| = N_L \|f\|_L$ for each $f \in F(T)$.

Clearly bounding sets exist for each $J \in F(T)'$.

DEFINITION 4. Let L be a compact subset of T and let $J \in F(T)'$. L is called a *vanishing set* for J if $\|f\|_L = 0$ implies $J(f) = 0$.

Of course a bounding set for a continuous linear functional is a vanishing set. We now prove the converse.

PROPOSITION 3. *If* L *is a vanishing set for a continuous linear functional* J *on* F(T), *then* L *is a bounding set for* J.

Proof. Suppose L is a vanishing set for J and let $f \in F(T)$. Let f^* denote the restriction of f to L and observe that all functions in F(L) can be extended to the "Stone-Cech" compactification $\beta(T)$ of T ([3, p. 8], [5], [7, p. 152]). Consider the map

$$J^* : F(L) \to F$$
$$f^* \to J(f).$$

J^* is seen to be well-defined because L is a vanishing set. It is evident that J^* is a linear functional on F(L). To see that J^* is continuous, let (f_s^*) be a net of functions in F(L) converging uniformly to O on L. By the Ellis-Tietze extension Theorem ([5]) we see that each f_s^* can be extended to $\beta(T)$ (and therefore to T) in such a way that the extension, denoted by f_s, satisfies

$$\sup_{t \in \beta(T)} |f_s(t)| = \|f_s\|_{\beta(T)} = \|f_s^*\|_L = \sup_{t \in T} |f_s(t)|.$$

Thus the net (f_s) converges uniformly to O on T; hence $f_s \to 0$ in the compact-open topology on F(T). As $J(f_s) = J^*(f_s^*)$ and J is continuous, we see that $J^*(f_s^*) \to 0$ so J^* is seen to be continuous. By Proposition 1 there exists a bounded measure μ^* on L such that $J^*(f^*) = J(f) = \int_L f^* \, d\mu^*$.

As $|J^*(f^*)| \leq \|f^*\|_L \|\mu\|$, it is seen that L is a bounding set for J.

DEFINITION 5. Let $J \in F(T)'$, and let $\mathfrak{D}_J$ denote the set of all $U \in \mathcal{S}$ such that $J(fk_U) = 0$ for each $f \in F(T)$. The set $F_J = C(\cup \mathfrak{D}_J)$ is called the *support of* J.

We show in the remainder of this section that associated with each $J \in F(T)'$ there is a bounded measure μ_J with compact support such that $F_J = F_{\mu_J}$, that the restriction μ_J^* of μ_J to F_J is a well-defined bounded measure and $J(f) = \int_{F_J} f^* \, d\mu_J^*$. Thus it follows that $F_J = F_{\mu_J}$ is a bounding set for J. It will also be seen that it is the unique minimal bounding set.

PROPOSITION 4. *Let* J *be a continuous linear functional on* F(T). *Then:* (a) *if* $U \in \mathfrak{D}_J$ *and* W *is a clopen subset of* U, *then* $W \in \mathfrak{D}_J$; (b) $\mathfrak{D}_J$ *is a ring of sets.*

Proof. To prove (*a*) we observe for any $f \in F(T)$ and any clopen subset W of $U \in \mathfrak{D}_J$ that $fk_W = fk_W k_U$. To prove (*b*) we first note that if $U_1, \cdots, U_n$ are pairwise disjoint clopen sets, then, letting $U = \cup U_i$, $k_U = \Sigma k_{U_i}$. Thus $\cup U_i \in \mathfrak{D}_J$. This, together with (*a*) implies (*b*).

PROPOSITION 5. *Let* $J \in F(T)'$. *Then:* (*a*) *the support* F_J *of* J *is compact,* $F_J = \varnothing$ *if and only if* J *is trivial;* (*b*) *if* $G \in \mathcal{S}$ *and* $G \cap F_J \neq \varnothing$, *then there exists* $f \in F(T)$ *such that* f *vanishes on* CG *and* $J(f) = 1$.

Proof. (*a*): If J is trivial then $\mathcal{S} = \mathfrak{D}_J$ and F_J is empty. Conversely suppose $F_J = \varnothing$ and that J is nontrivial. Then there exists a compact set L which is a bounding set for J and $g \in F(T)$ such that $J(g) \neq 0$. As $F_J = \varnothing$, $\cup \mathfrak{D}_J = T$ so there exist $U_1, \cdots, U_n \in \mathfrak{D}_J$ such that $L \subset \cup U_i = W$. As L is a bounding set for J, L is a vanishing set. Thus $CW \in \mathfrak{D}_J$. By Proposition 4 (*b*) then $T \in \cup \mathfrak{D}_J$. Therefore J is trivial.

To show that F_J is compact we observe that if $U \subset CL$ and $U \in \mathcal{S}$, then $U \in \mathfrak{D}_J$ because L is a vanishing set. Hence $CL \subset \cup \mathfrak{D}_J$ and $F_J \subset L$.

(*b*): For $G \in \mathcal{S}$, if $F_J \cap G \neq \varnothing$, then $G \notin \mathfrak{D}_J$. Thus there is some $g \in F(T)$ such that $J(gk_G) \neq 0$. Now let $f = J(gk_G)^{-1} gk_G$.

LEMMA 1. *Let* $J \in F(T)'$ *and suppose that any clopen set* W *containing* F_J *has the property that whenever* $f \in F(T)$ *vanishes on* W *then* $J(f) = 0$. *Then* F_J *is a vanishing set for* J.

Proof. Let f vanish on F_J and let $A_n = \{t \in T \mid |f(t)| < 1/n\}$. As $A_n \in \mathcal{S}$ for each n and $F_J \subset A_n$, then $J(f) = J(fk_{A_n}) + J(fk_{CA_n}) = J(fk_{A_n})$. As $|J(fk_{A_n})| \leq N_L \|fk_{A_n}\|_L < N_L(1/n)$ for some compact set $L \subset T$ and any $n > 0$, $J(f) = 0$.

LEMMA 2. *If* $J \in F(T)'$ *and* T *is a Lindelöf space, then* F_J *is a vanishing set.*

Proof. Let F_J be a subset of a clopen set W. Since CW is closed and $CW \subset \cup \mathfrak{D}_J$, there exist $U_1, \cdots, \in \mathfrak{D}_J$ such that $CW \subset \bigcup_{i=1}^{\infty} U_i$. Since $\mathfrak{D}_J$ is a ring of sets, we may assume that the family (U_i) is pairwise disjoint. Thus

$CW = \bigcup_{i=1}^{\infty} (U_i \cap CW)$ and, letting $U_i \cap CW = V_i$ for each i, we see that each $V_i \in \mathfrak{D}_J$. Hence $k_{CW} = \sum_{i=1}^{\infty} k_{V_i}$ where the convergence of the series is pointwise. If L is any compact subset of T we see that there is some $N > 0$ such that $L \cap CW = L \cap \left(\bigcup_{i=1}^{n} V_i\right) = \bigcup_{i=1}^{n} (L \cap V_i)$ for any $n \geq N$. Thus

$$\sup_{t \in L} \left| k_{CW} - \sum_{i=1}^{n} k_{V_i} \right| = 0 \qquad \text{for} \quad n \geq N$$

and the series is seen to converge in the compact-open topology. If f vanishes on W then $J(f) = J(fk_{CW}) = J\left(f \sum_{i=1}^{\infty} k_{V_i}\right) = \sum_{i=1}^{\infty} J(fk_{V_i}) = 0$. The desired result now follows from Lemma 1.

LEMMA 3. *If* $J \in F(T)'$ *and* T *is a compact space, then* F_J *is a vanishing set.*

Proof. We show that if F_J is a subset of a clopen set W and f vanishes on W, then $J(f) = 0$. By Lemma 1 it will then follow that F_J is a vanishing set.

To do this let F_J be a subset of the clopen set W. Since W is clopen and therefore CW is clopen and compact, there exist $V_i \in \mathfrak{D}_J$, $i = 1, 2, \cdots, n$, such that $CW = \bigcup_{i=1}^{n} V_i$. As $\mathfrak{D}_J$ is a ring of sets, $CW \in \mathfrak{D}_J$. Hence if f vanishes on W, then $J(f) = J(fk_{CW}) = 0$.

We now prove that if T is any O–dimensional space and $J \in F(T)'$, F_J is a vanishing set for J. This property and the properties of F_J proved in Proposition 5 are the properties of support which made the applications in [3] possible; in particular these properties enabled the Authors in [3, Sec. 3, Theorem 3] to obtain a generalization of [9, Theorem 1].

THEOREM 2. *If* $J \in F(T)'$ *and* μ_J *is the bounded measure defined by taking* $\mu_J(U) = J(k_U)$ *for each* $U \in \mathfrak{S}$, *then* $F_{\mu_J} = F_J$ *and* F_J *is the minimal (with respect to set-inclusion) vanishing set for* J.

Proof. Let L be any vanishing set for J and consider the bounded measure μ_J^* on $\mathfrak{S} \cap L$ defined by taking $\mu_J^*(L \cap U) = \mu_J(U) = J(k_U)$ for any $U \in \mathfrak{S}$. For $U, V \in \mathfrak{S}$, if $U \cap L = V \cap L$, then $(U - U \cap V) \cap L = \varnothing$ and, since L is a vanishing set, $J(k_{U-U\cap V}) = 0$. Thus $J(k_U) = J(k_{U \cap V}) = J(k_V)$ and μ_J^* is seen to be well-defined. We see from the proofs of Propositions 1 and 3 that $J^*(f^*) = J(f) = \int_L f^* \, d\mu_J^*$, where f^* denotes the restriction of $f \in F(T)$ to the vanishing set L.

We next show that $F_J = F_{J^*}$, $F_{\mu_J} = F_{\mu_J^*}$ and $F_{\mu_{J^*}} = F_{J^*}$. Once this has been done then, since $\mu_{J^*} = \mu_J^*$ by definition and L is compact, Lemma 2 or Lemma 3 implies that

$$J^*(f^*) = J(f) = \int_L f^* \, d\mu_J^* = \int_{F_{J^*}} f^* \, d\mu_J^* = \int_{F_J} f^* \, d\mu_J^* .$$

It then follows that F_J is a vanishing set for J.

To show that $F_J = F_{J^*}$ we note that $J(fk_U) = J^*(fk_{U \cap L})$ and therefore $U \in \mathfrak{D}_J$ if and only if $U \cap L \in \mathfrak{D}_{J^*}$. Since $F_J \subset L$ (see proof of Proposition 5) it follows that $C(\cup \mathfrak{D}_J) = C(\cup \mathfrak{D}_{J^*})$.

To show that $F_{\mu_J} = F_{\mu_J^*}$ we simply note that $\mu_J(U) = \mu_J^*(U \cap L)$ and therefore $U \cap L \in \mathfrak{D}_{\mu_J^*}$ if and only if $U \in \mathfrak{D}_{\mu_J}$.

To show that $F_{\mu_J^*} = F_{J^*}$, note that if $U \cap L \in \mathfrak{D}_{J^*}$, then $J^*(k_{W \cap L}) = J^*(k_{U \cap L}) = 0$ for any $W \cap L \subset U \cap L$. Hence $\mu_J^*(W \cap L) = \mu_J^*(U \cap L) = 0$ for all $W \cap L \subset U \cap L$ and therefore $U \cap L \in \mathfrak{D}_{\mu_J^*}$. Conversely if $U \cap L \in \mathfrak{D}_{\mu_J^*}$ then, to compute $\int_L f^* k_{U \cap L} \, d\mu_J^*$ we need only consider partitions $(U_i \cap L)$, $i = 1, 2, \cdots, n$, of L such that $U_i \cap L$ is a subset of $U \cap L$ or $U_i \cap L$ is a subset of $CU \cap L$ for each i, $1 \leqq i \leqq n$. For such partitions the associated partial sums will be zero. Thus $J^*(f^* k_{U \cap L}) = \int_L f^* k_{U \cap L} \, d\mu_J^* = 0$ and $U \cap L \in \mathfrak{D}_{J^*}$. Hence $\mathfrak{D}_{J^*} = \mathfrak{D}_{\mu_J^*}$ and $F_{J^*} = F_{\mu_J^*}$.

The minimality and uniqueness of F_J follows from Proposition 5 (*b*).

In [3] we proved the following Theorem: Let F(T) be the algebra of continuous functions mapping the O–dimensional Hausdorff space T into the complete discrete field F with compact-open topology, and let F_J be a vanishing set for each $J \in F(T)'$. Then F(T) is F–barreled if and only if for every $E \subset T$ which is closed but not compact there is some $f \in F(T)$ which is unbounded on E. We observe that Theorem 2 shows this result to be true for O–dimensional Hausdorff spaces T [1].

Consider the following question: "Given any bounded measure μ with compact support on T, is there a continuous linear functional $J \in F(T)'$ associated with μ such that $J(k_U) = \mu(U)$ for each $U \in \mathcal{S}$, so that $J(f) = \int_{F_\mu} f^* \, d\mu^* = \int_{F_J} f^* \, d\mu^*$?". A natural way to begin to approach this question is by considering the restriction μ^* of μ to $F_\mu \cap \mathcal{S}$. Then define J by taking $J(f) = \int_{F_\mu} f^* \, d\mu^*$ where f^* denotes the restriction of $f \in F(T)$ to F_μ. By previous arguments J is a well-defined continuous linear functional on F(T) into F if $\mu^*(U \cap F_\mu) = \mu(U)$ is a well-defined

(1) It has come to the Authors' attention that this result was proved by R. L. Ellis using other techniques in his doctoral dissertation (Duke University, 1966) for spherically complete fields.

bounded measure on $F_\mu \cap \mathcal{S}$. Conversely if the desired continuous linear functional J exists, then the map μ defined on $F_\mu \cap \mathcal{S}$ for each $U \in \mathcal{S}$ by $\mu^*(U \cap F_\mu) = \mu^*(U \cap F_J) = \mu(U) = J(k_U)$ is well-defined by Theorem 2. Thus, in answer to the original question, we may say: "There is a $J \in F(T)'$ such that $J(k_U) = \mu(U)$ for each $U \in \mathcal{S}$ if and only if the map μ^* defined on $F_\mu \cap \mathcal{S}$ by $\mu^*(U \cap F_\mu) = \mu(U)$ for each $U \in \mathcal{S}$ is well-defined". Example 2 of the next section will show that there are bounded measures with empty support and bounded measures with nonempty compact support whose restrictions to $F_\mu \cap S$ are not well-defined. Clearly such measures cannot correspond to continuous linear functionals in the manner described above. We note also that the correspondence $\mu \to J$ so obtained is injective because the linear span $X = [\{k_U \mid U \in \mathcal{S}\}]$ of the characteristic functions of the clopen sets is dense in F(T); thus if J and J′ are continuous linear functionals and $J(k_U) = J'(k_U)$ for each $U \in \mathcal{S}$, J and J′ must agree on all of F(T). In summary we have:

THEOREM 3. *Let* $B(T)^c$ *denote the linear space of bounded measures* μ *with compact nonempty support* F_μ *such that the restriction* μ^* *of* μ *to* $F_\mu \cap \mathcal{S}$ *(as defined above) is well-defined. Then* $B(T)^c$ *is linearly isomorphic to* $F(T)'$.

As has already been noted, a bounded measure μ with nonempty compact support on T need not determine a continuous linear functional on F(T). Such a bounded measure determines a linear functional J on $X = [\{k_U \mid U \in \mathcal{S}\}]$ via $J(\Sigma\alpha_i k_{U_i}) = \Sigma\alpha_i \mu(U_i)$, however. If J is continuous on this dense subspace of F(T), J may be extended (by continuity) to a continuous linear functional J′ on F(T). Since, for any $U \in \mathcal{S}$, $J'(k_U) = J(k_U) = \mu(U)$, Theorem 2 implies that the restriction μ^* of μ to $F_\mu \cap \mathcal{S}$ is well-defined. Thus if $\mu \notin B(T)^c$, J cannot be continuous.