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Fixed points in complete metric spaces

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Matematica. — *Fixed points in complete metric spaces* (*). Nota di SIMEON REICH, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Vengono stabilite varie proposizioni che forniscono condizioni sufficienti per l'esistenza di punti fissi, relativi a funzioni condensatrici a più valori su di uno spazio metrico, completo e limitato.

1. SOME DEFINITIONS

Let (X, d) be a metric space. In the sequel use will be made of the following notations:

$P(X) = \{A \mid A$ is a nonempty subset of $X\}$,
$BN(X) = \{A \mid A$ is a nonempty bounded subset of $X\}$,
$CL(X) = \{A \mid A$ is a nonempty closed subset of $X\}$,
$CB(X) = \{A \mid A$ is a nonempty closed and bounded subset of $X\}$,
$C(X) = \{A \mid A$ is a nonempty compact subset of $X\}$.

A nonnegative function m defined on $BN(X)$ will be called a measure of noncompactness if it enjoys the following two properties:

(i) $m(A) = 0 \Longleftrightarrow A$ is totally bounded,
(ii) $m(A) = 0 \Rightarrow m(A \cup B) = m(B)$,

where A and B belong to $BN(X)$.

Here are three examples of such measures.

(i) $m(A) = 0$ if A is totally bounded, $m(A) = 1$ otherwise,
(ii) $a(A) = \inf\{r > 0 \mid A$ can be covered by a finite number of subsets of X of diameter less than $r\}$ (see [4], p. 303),
(iii) $b_X(A) = \inf\{r > 0 \mid A$ can be covered by a finite number of balls with centers in X and radius $r\}$.

Here a ball with center y and radius r is the set

$$B_X(y; r) = \{x \in X \mid d(y, x) \leq r\}.$$

A real function q defined on $X \times X$ equipped with the cartesian product topology will be called a nearness function if it is lower semicontinuous.

Here are three examples of such functions.

(i) $q(x, y) = 0$ if $x = y$, $q(x, y) = 1$ otherwise,
(ii) d, the given metric,
(iii) any metric on X which induces a coarser topology than the one generated by d.

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If A and B are nonempty subsets of X, we put

$$Q(A, B) = \inf\{q(a, b) \mid a \in A, b \in B\},$$

$$H_q(A, B) = \max[\sup\{Q(a, B) \mid a \in A\}, \sup\{Q(A, b) \mid b \in B\}].$$

(In the last definition x is identified with $\{x\}$).

In general, both Q and H_q may take the values $+\infty$ and $-\infty$. If $q = d$ we shall write D and H for Q and H_q respectively. On CB(X) H is actually a metric—the Hausdorff metric.

Let $F: X \to P(X)$ be a multi-valued function. We associate with it a function $G_F: P(X) \to P(X)$ by defining $G_F(A) = \cup\{F(a) \mid a \in A\}$, where $A \in P(X)$.

F will be called m–condensing, m being a measure of noncompactness, if it satisfies

$$(1) \qquad A \in BN(X) \wedge m(A) > 0 \Rightarrow m(G_F(A)) < m(A).$$

It will be called q–contractive, q being a nearness function, if it satisfies

$$(2) \qquad x \neq y \Rightarrow H_q(F(x), F(y)) < q(x, y).$$

2. A FIXED POINT THEOREM

THEOREM 1 (Cf. [2], p. 506). *Let (X, d) be a bounded complete metric space. If a continuous $F: (X, d) \to (CB(X), H)$ is m–condensing as well as q–contractive, then it has a fixed point.*

Proof. Let $x \in X$ and consider $A = \{x\} \cup \{G_F^n(x) \mid n = 1, 2, \cdots\}$. Clearly $A = \{x\} \cup G_F(A)$. Hence $m(A) = G_F(A) = 0$. Since X is complete, the closure of A, which will be denoted by K, is compact. It is not difficult to see that the continuity of F implies the invariance of K under G_F. Define a real function p on K by $p(y) = Q(y, F(y))$, $y \in K$. p is lower semicontinuous. Therefore $p(z) = \inf\{p(y) \mid y \in K\}$ for some $z \in K$. There exists $c \in F(z)$ such that $p(z) = q(z, c)$. If $c \neq z$, then $p(c) = Q(c, F(c)) \leq \leq H_q(F(z), F(c)) < q(z, c)$ – a contradiction. Consequently, $c = z$ is a fixed point of F.

3. AN APPLICATION

In order to present an application of this Theorem we need several preliminary notions.

Let k be nonnegative. A multi-valued function $F: X \to P(X)$ will be called a k–set–contraction with respect to a measure of noncompactness m if it satisfies

$$(3) \qquad m(G_F(A)) \leq km(A) \quad , \quad A \in BN(X).$$

Let m be fixed. If F is a k–set–contraction, we put $m(F) = \inf\{k \mid F \text{ is a } k\text{–set–contraction}\}$.

Assume now that $F : X \to C(X)$. Then $G_F^n : C(X) \to C(X)$ for every $n \geq 1$ ([6], p. 157). Therefore we can define $F^n : X \to C(X)$ by $F^n(x) = G_F^n(x)$. Note that $G_{F^n} = G_F^n$. If, in addition, F is a k–set–contraction, F^n is a k^n–set–contraction. One can see that $r(F) = \lim_{n\to\infty} [m(F^n)]^{1/n}$ exists and equals $\inf\{[m(F^n)]^{1/n} \mid n = 1, 2, \cdots\}$ (Cf. [8], p. 476).

THEOREM 2. *Let* (X, d) *be a bounded complete metric space. Let a* k*–set–contraction* $F : (X, d) \to (C(X), H)$ *be continuous and* q*–contractive. If* $r(F) < 1$, *then* F *has a fixed point.*

Proof. For any $0 < z < 1/r(F)$ define a new measure of noncompactness by $m_z(A) = \sum_{i=0}^{\infty} m(G_{F^i}(A))\, z^i$, $A \subset X$. Since $m_z(G_F(A)) \leq m_z(A)/z$, F is m_z–condensing for z sufficiently near $1/r(F)$. The result now follows by Theorem 1.

COROLLARY 1. *Let* (X, d) *be a bounded complete metric space, and let* $F : X \to C(X)$ *be* d*–contractive. If for some natural* n

$$H(F^n(x), F^n(y)) \leq kd(x, y), \tag{4}$$

where $0 \leq k < 1$, *then* F *has a fixed point.*

This is a partial generalization of a result due to Nadler, Jr. ([7], p. 479).

4. ANOTHER FIXED POINT THEOREM

A generalized metric has all the properties of an ordinary metric except that it may be infinite (see [5], p. 541). For example, if (X, d) is a metric space, $(CL(X), H)$ is a generalized metric space.

THEOREM 3 (Cf. [3], p. 465). *Let* (X, d) *be a complete metric space. Let the continuous* $F : (X, d) \to (CL(X), H)$ *be* b_X*–condensing. If there exists a bounded sequence* $\{x_n\}_{n=1}^{\infty} \subset X$ *such that* $D(x_n, F(x_n)) \xrightarrow[n\to\infty]{} 0$, *then* F *has a fixed point.*

Proof. Put $A = \{x_n\}_{n=1}^{\infty}$ and consider, for any positive ε, the set $B = \{z \in X \mid D(z, G_F(A)) < \varepsilon\}$. There is a natural number N which satisfies $\{x_n\}_{n=N}^{\infty} \subset B$. It follows that $b_X(A) \leq b_X(B) \leq b_X(G_F(A)) + \varepsilon$. Hence $b_X(A) \leq b_X(G_F(A))$, so that the closure of A is compact. Let $\{y_n\}_{n=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ be a convergent subsequence with limit y. y is a fixed point of F because $D(y, F(y)) \leq d(y, y_n) + D(y_n, F(y)) \leq d(y, y_n) + D(y_n, F(y_n)) + H(F(y_n), F(y)) \xrightarrow[n\to\infty]{} 0$.

Observe that "b_X–condensing" can be replaced by "a–condensing" in the statement of the Theorem.

COROLLARY 2 (Cf. [7], p. 484). *Let* (X, d) *be a bounded complete metric space, let* $F_0 : (X, d) \to (CL(X), H)$ *be continuous and* b_X*–condensing, and let* $F_n : X \to CL(X)$ *have a fixed point* x_n *for each* $n \geq 1$. *If the sequence* $\{F_n\}_{n=1}^{\infty}$ *converges uniformly to* F_0, *then a subsequence of* $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* F_0.

5. AN APPLICATION

THEOREM 4. *Let* (X, d) *be a bounded complete metric space. Suppose that* $k : (0, \infty) \to [0, 1)$ *and that for positive* r $\limsup_{t \to r} k(t) < 1$. *If* $F : X \to C(X)$ *satisfies*

$$\text{(5)} \qquad H(F(x), F(y)) \leq k(d(x, y))\, d(x, y),$$

where $x \neq y$, *then it has a fixed point.*

Proof. Consider an $A \subset X$ with $b_X(A) = R > 0$. Positive $e(R)$ and $S(R) < 1$ can be found such that $k(t) \leq S$ for all $R - e \leq t \leq R + e$ and $R - e \leq (R + e) S = r < R$. Since $A \subset \bigcup_{i=1}^{n} B_X(x_i ; R + e)$ where $\{x_i\}_{i=1}^{n}$ is a finite subset of X, this means that $G_F(A) \subset \bigcup_{i=1}^{n} B_{C(X)}(F(x_i) ; r)$. $F(x_i) \subset \bigcup_{j=1}^{n_i} B_X(y_j^{(i)} ; 1/2(R - r))$ for some finite subset $\{y_j^{(i)}\}_{j=1}^{n_i}$ of X, $1 \leq i \leq n$. It follows that $G_F(A) \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} B_X(y_j^{(i)} ; 1/2(R+r))$, so that F is b_X-condensing. Let $x_0 \in X$ and let $x_1 \in F(x_0)$. We can assume that $x_1 \neq x_0$. Choose $x_2 \in F(x_1)$ such that $d(x_1, x_2) \leq H(F(x_0), F(x_1)) \leq k(d(x_0, x_1))\, d(x_0, x_1)$. In this manner, assuming that $x_{n+1} \neq x_n$, we can construct inductively a sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies $x_n \in F(x_{n-1})$ and $d(x_n, x_{n+1}) \leq$ $\leq k(d(x_{n-1}, x_n))\, d(x_{n-1}, x_n) < d(x_{n-1}, x_n)$, $n = 1, 2, \cdots$. Where $L = \lim_{n \to \infty} d(x_n, x_{n+1})$ positive, one would obtain $L < \limsup_{t \to L} k(t) L < L$, an impossible situation. Hence $L = 0$. At this point, an appeal to Theorem 3 yields the desired conclusion.

This proposition can be considered a generalization of a result of Browder's ([1], p. 28).

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