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Some Topological Considerations in General Relativity

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Geometria. — *Some Topological Considerations in General Relativity.* Nota (*) di JOHN PORTER e ALAN THOMPSON, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — In una varietà compatta Einstein–Lorentziana, definita nell'introduzione, si studiano, nell'ambito della relatività generale, le conseguenze del fatto che la caratteristica di Eulero–Poincaré si annulla. In particolare: si studiano il tensore C di Weyl e il tensore T, energia momento; si dimostra che soltanto una classe ristretta di varietà ammette un campo di fluido perfetto o un campo elettromagnetico non nullo, e si danno condizioni necessarie e sufficienti perché la varietà ammetta un campo gravitazionale « in vacuo ».

1. INTRODUCTION

By an Einstein-Lorentz manifold we will mean a four-dimensional connected orientable, C^∞-differentiable manifold M carrying a pseudo-Riemannian structure g of signature 2 and rank 4 at each point. If in addition M is compact, then we have the well known result [1], [2] that $\chi(M)$, the Euler-Poincaré characteristic of M, vanishes.

We discuss here the consequences of this result for the general theory of relativity, in particular we will be concerned with the Weyl conformal curvature tensor C and the energy-momentum tensor T of a compact Einstein-Lorentz manifold (1). We show that only a restricted class of manifolds can support a perfect-fluid or non-null electromagnetic field and give a necessary condition for M to admit a vacuum solution of the field equations.

Some results concerning the Pontrjagin number of M are also given.

2. PRELIMINARY RESULTS

Consider the domain of a local chart on a compact Einstein-Lorentzian manifold M, with corresponding local coordinates $\{x^a\}$ $a = 1, \cdots, 4$. Let $R^a{}_{bcd}$, g_{ab} and ε_{abcd} ($\varepsilon_{1234} = |\mathrm{Det}\, g_{ab}|^{1/2}$) be the local expressions for the curvature tensor, the metric and the Levi-Civita alternating tensor, respectively. We introduce the "dual" with respect to a pair of anti-symmetric indices by

$$S^*_{ab\cdots k} = \frac{1}{2}\, \varepsilon_{ab}{}^{pq}\, S_{pq\cdots k},$$

(*) Pervenuta all'Accademia il 28 settembre 1971.

(1) J. Zund [3] has also attempted the discussion of similar questions. However he fails to exploit the vanishing of the Euler-Poincaré characteristic, and his discussion of "physical schemes" is not valid.

for an arbitrary tensor $S_{ab\cdots k}$, antisymmetric in the index pair ab, and it follows that

$$S^{**}_{ab\cdots k} = - S_{ab\cdots k}.$$

In particular for the curvature tensor we introduce;

$$({}^*R)_{abcd} = R^*_{abcd} = \frac{1}{2} \varepsilon_{ab}{}^{pq} R_{pqcd},$$

$$(R^*)_{abcd} = R_{ab}{}^*_{cd} = \frac{1}{2} R_{abpq} \varepsilon^{pq}{}_{cd},$$

$$({}^*R^*)_{abcd} = R^{*\,*}_{abcd} = \frac{1}{4} \varepsilon_{ab}{}^{pq} R_{pqrs} \varepsilon^{rs}{}_{cd}.$$

As is well known the curvature tensor R_{abcd} admits the decomposition [4], [5]

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{r}{12} g_{abcd}, \tag{2.1}$$

where C_{abcd} is the Weyl conformal curvature and,

$$\begin{aligned} g_{abcd} &= 2 g_{a[c} g_{d]b} \quad , \quad r = R^b{}_b = R^{ab}{}_{ab}, \\ E_{abcd} &= - g_{abe[c} S^e{}_{d]} \quad , \quad S^e{}_d = R^e{}_d - \frac{1}{4} r \delta^e{}_d. \end{aligned} \tag{2.2}$$

The tensors C_{abcd}, E_{abcd} and g_{abcd} possess the same symmetries as the curvature tensor and satisfy:

$$\begin{aligned} ({}^*C)_{abcd} &= C^*_{abcd} = C_{ab}{}^*_{cd} = (C^*)_{abcd} \Rightarrow {}^*C^* = - C, \\ ({}^*E)_{abcd} &= E^*_{abcd} = - E_{ab}{}^*_{cd} = - (E^*)_{abcd} \Rightarrow {}^*E^* = E, \\ ({}^*g)_{abcd} &= g^*_{abcd} = g_{ab}{}^*_{cd} = (g^*)_{abcd} \Rightarrow {}^*g^* = - g. \end{aligned} \tag{2.3}$$

(We also define $\overset{*}{C}_{abcd} = ({}^*C)_{abcd} = (C^*)_{abcd}$).

From these identities we immediately deduce

$$\begin{aligned} E_{abcd} &= \frac{1}{2} (R_{abcd} + ({}^*R^*)_{abcd}), \\ C_{abcd} + \frac{r}{12} g_{abcd} &= \frac{1}{2} (R_{abcd} - ({}^*R^*)_{abcd}). \end{aligned} \tag{2.4}$$

In terms of a quasi-orthonormal tetrad, [5] $\{k, m, \bar{t}, t\}$ (k^a and m^a real null vectors, t^a a complex null vector, $k_a m^a = \bar{t}_a t^a = 1$, all other products zero) we construct the bivectors

$$\begin{aligned} V_{ab} &= 2 k_{[a} \bar{t}_{b]} \quad , \quad U_{ab} = 2 m_{[a} t_{b]}, \\ M_{ab} &= 2 k_{[a} m_{b]} + 2 \bar{t}_{[a} t_{b]}, \end{aligned} \tag{2.5}$$

with

$$M_{ab} M^{ab} = - 2 V_{ab} U^{ab} = - 4, \qquad \text{all other such contractions zero.}$$

(V, U and M are " anti-self dual " in the sense $\overset{*}{V} = - iV$, etc.).

We define the "complexified Weyl tensor"

$$P_{abcd} = C_{abcd} + i\overset{*}{C}_{abcd},$$

for which we note

$$\overset{*}{P}_{abcd} = -iP_{abcd} \quad ; \quad \frac{1}{2} P_{abcd} P^{abcd} = C_{abcd} C^{abcd} + iC_{abcd} \overset{*}{C}{}^{abcd},$$

and the expansion (indices suppressed)

$$\text{(2.6)} \qquad P = C^{(5)} UU + C^{(4)} (MU + UM) + C^{(3)} (MM + UV + VU) + \\ + C^{(2)} (MV + VM) + C^{(1)} VV.$$

For future reference we note the Pirani-Petrov classification of the Weyl tensor in the following form:

C_{abcd} of Type I $\Rightarrow \exists V, U, M \ni C^{(5)} = 0, C^{(4)} \neq 0,$

C_{abcd} of Type II $\Rightarrow \exists V, U, M \ni C^{(5)} = C^{(4)} = 0, C^{(3)} \neq 0,$

C_{abcd} of Type D $\Rightarrow \exists V, U, M \ni C^{(A)} = 0, \forall A, A \neq 3, C^{(3)} \neq 0,$

C_{abcd} of Type III $\Rightarrow \exists V, U, M \ni C^{(5)} = C^{(4)} = C^{(3)} = 0, C^{(2)} \neq 0,$

C_{abcd} of Type N $\Rightarrow \exists V, U, M \ni C^{(A)} = 0, \forall A, A \neq 1, C^{(1)} \neq 0.$

From (2.1) and the identifications (2.3) we have

$$({}^*R^*)_{abcd} R^{abcd} = \left(-C + E - \frac{r}{12} g\right) : \left(C + E + \frac{r}{12} g\right),$$

(where we employ the shorthand $R : R = R_{abcd} R^{abcd}$ etc.) and hence

$$\text{(2.7)} \qquad ({}^*R^*) : R = -C : C + E : E - \frac{r^2}{6},$$

since $C^{ab}{}_{ab} = E^{ab}{}_{ab} = 0$.
Now

$$E : E = E_{abcd} E^{abcd} = g_{abe[c} S^e{}_{d]} g^{abfc} S^d{}_f = 2\left(R^{ab} R_{ab} - \frac{1}{4} r^2\right);$$

thus (2.6) becomes

$$\text{(2.7 a)} \qquad ({}^*R^*) : R = -C_{abcd} C^{abcd} + 2\left(R^{ab} R_{ab} - \frac{1}{3} r^2\right).$$

From the results (2.3) and the symmetries of C_{abcd}, E_{abcd}, and g_{abcd}, we have

$$\overset{*}{C}_{abcd} E^{abcd} = C_{abcd} \overset{*}{E}{}^{abcd} = -C_{abcd} \overset{*}{E}{}^{cdab} = -C_{cdab} \overset{*}{E}{}^{cdab} \\ = -\overset{*}{C}_{abcd} E^{abcd},$$

and hence

$$\text{(2.8)} \qquad \overset{*}{C}_{abcd} E^{abcd} = 0.$$

Similarly we have

$$(^*E)_{abcd}\,E^{abcd} = (E^*)_{abcd}\,E^{abcd} = 0\,,$$
$$(2.9)\qquad (^*E)_{abcd}\,g^{abcd} = (E^*)_{abcd}\,g^{abcd} = 0\,,$$
$$\overset{*}{g}_{abcd}\,g^{abcd} = \varepsilon_{abcd}\,g^{abcd} = 0\,.$$

As an immediate consequence of (2.8) and (2.9) we have

$$(2.10)\qquad (^*R)_{abcd}\,R^{abcd} = (R^*)_{abcd}\,R^{abcd} = \overset{*}{C}_{abcd}\,C^{abcd}.$$

3. Consequences of the Vanishing of the Euler-Poincaré Characteristic $\chi(M)$

Avez [6] has proved that the Euler-Poincaré characteristic of a compact orientable pseudo-riemannian 4–manifold M [(2)] is given by

$$(3.1)\qquad \chi(M) = \frac{(-1)^{[p/2]}}{8\pi^2}\int_M (^*R^*)_{abcd}\,R^{abcd}\,dv\,.$$

Hence if we restrict the above to compact Einstein-Lorentz manifolds M (for which $\chi(M)$ vanishes) we have with equation (2.6 a)

$$(3.1\ a)\qquad \int_M C_{abcd}\,C^{abcd}\,dv = 2\int_M \left(R^{ab}\,R_{ab} - \frac{1}{3}\,r^2\right)dv\,.$$

For a vacuum gravitational field, $R_{ab} = 0\,(= r)$ and consequently $R_{abcd} = C_{abcd}$, and we have from (3.1 a):

THEOREM 1. *A necessary condition for a compact Einstein–Lorentz manifold* M *to represent a vacuum gravitational field is*

$$\int_M R_{abcd}\,R^{abcd}\,dv = 0\,.$$

COROLLARY. *If* M *represents a vacuum gravitational field and* $R_{abcd}R^{abcd} \not\equiv 0$ *at every point of* M, *then* $R_{abcd}\,R^{abcd}$ *changes sign on* M.

The field equations of general relativity are (in suitably chosen units)

$$(4.1)\qquad G_{ab} = R_{ab} - \frac{1}{2}\,r g_{ab} = -KT_{ab}\,,$$

(2) We consider the pseudo-Riemannian metric having p positive squares and $(4-p)$ negative squares.

where T_{ab} is the energy-momentum tensor, and K is the Einstein gravitational constant. For a perfect fluid we have

$$T_{ab} = (\mu + p)\, u_a\, u_b + p g_{ab},$$

where μ is the density, p the pressure and u^a is the unit (time-like) tangent vector field to the streamlines.

From (4.1) we have (with $T = g^{ab} T_{ab}$)

$$\left(R^{ab} R_{ab} - \frac{r^2}{3}\right) = K^2 \left(T^{ab} T_{ab} - \frac{T^2}{3}\right),$$

and hence for a perfect fluid

$$\left(R^{ab} R_{ab} - \frac{r^2}{3}\right) = K^2 \left(\frac{2}{3}\mu^2 + p^2 + 2\mu p\right) > 0. \tag{4.2}$$

For an electromagnetic field, F_{ab}, we have

$$T_{ab} = \frac{1}{4} g_{ab} F_{cd} F^{cd} - F_{ac} F_b{}^c,$$

and hence

$$\left(R^{ab} R_{ab} - \frac{1}{3} r^2\right) = K^2 \left(F_{ac} F^{bc} F^{ad} F_{bd} - \frac{1}{4} (F_{cd} F^{cd})^2\right). \tag{4.3}$$

But for arbitrary bivectors X_{ab} and Y_{ab} we have the identity [3]

$$X_{ac} Y^{bc} - \overset{*}{X}_{ac} \overset{*}{Y}{}^{bc} = \frac{1}{2} \delta_a^b X_{cd} Y^{cd}, \tag{4.4}$$

and hence (put $Y^{bc} = \overset{*}{X}{}^{bc}$):

$$X_{ac} \overset{*}{X}{}^{bc} = \frac{1}{4} \delta_a^b X_{cd} \overset{*}{X}{}^{cd}. \tag{4.5}$$

From (4.5) we deduce

$$F_{ab} F^{bc} F^{ad} F_{bd} = \left(\frac{1}{2} \delta_a^b F_{de} F^{de} + \overset{*}{F}_{ad} \overset{*}{F}{}^{bd}\right) (F^{ac} F_{bc})$$

$$= \frac{1}{2} (F_{cd} F^{cd})^2 + (F_{ac} \overset{*}{F}{}^{ad}) (F_{bd} \overset{*}{F}{}^{bc}),$$

and consequently with the use of (4.5) in (4.3):

$$R^{ab} R_{ab} - \frac{1}{3} r^2 = \frac{1}{4} \varkappa^2 \{(F_{ab} F^{ab})^2 + (F_{ab} \overset{*}{F}{}^{ab})^2\} \geq 0, \tag{4.6}$$

for an electromagnetic field, with equality being achieved only if F_{ab} is a null, simple bivector (i.e. it represents a null electromagnetic field).

From the expansion (2.6) we have

$$C_{abcd}C^{abcd} + iC_{abcd}\overset{*}{C}{}^{abcd} = 2\,C^{(1)}\,C^{(5)} - 8\,C^{(2)}\,C^{(4)} + 12\,(C^{(3)})^2, \tag{4.7}$$

and consequently for Pirani-Petrov types III and N

$$C_{abcd}C^{abcd} = C_{abcd}\overset{*}{C}{}^{abcd} = 0. \tag{4.8}$$

With equation (3.1 a) we have:

THEOREM 2. *If a compact Einstein-Lorentz manifold* M *is of Pirani-Petrov type III or* N *(or conformally flat), then*

$$\int_M \left(R_{ab}\,R^{ab} - \frac{1}{3}\,r^2\right) dv = 0.$$

From the equations (4.2) and (4.6) we immediately deduce:

COROLLARY 1. *Compact Einstein-Lorentz manifolds of Pirani-Petrov types III and* N *(or conformally flat) admit neither perfect fluids nor non-null electromagnetic fields.*

5. THE PONTRJAGIN NUMBER p(M).

The Pontrjagin number of a compact Einstein-Lorentz manifold is given by [7]

$$p(M) = -\frac{1}{16\,\pi^2}\int_M (^*R)_{abcd}\,R^{abcd}\,dv,$$

and hence from (2.10) we deduce

$$p(M) = -\frac{1}{16\,\pi^2}\int_M \overset{*}{C}_{abcd}\,C^{abcd}\,dv. \tag{5.1}$$

THEOREM 3. *Let* M *be a compact Einstein-Lorentz manifold; then if* M *is of Pirani-Petrov type III or* N *(or conformally flat),* $p(M) = 0$.

Proof. Immediate from (4.7) and (5.1).

In conclusion we deal with some generalisations of these results to Pirani-Petrov type II, (or type D which we here regard as a special case of II).

Suppose that M is of type II (or D) with $C_{abcd}\overset{*}{C}{}^{abcd} = 0$, then there exists a basis $\{V, U, M\}$ such that

$$\frac{1}{2}\,P_{abcd}\,P^{abcd} = 12\,\{C^{(3)}\}^2 = C_{abcd}\,C^{abcd}.$$

For $C^{(3)} = \alpha + i\beta$, this implies $\alpha\beta = 0$, and

$$C_{abcd}\, C^{abcd} = 12\,(\alpha^2 - \beta^2).$$

For type II we distinguish the subclasses

II $a \Rightarrow \exists$ a basis $\{V, U, M\} \ni C^{(5)} = C^{(4)} = 0\,,\; C^{(3)} = \text{real} \neq 0$.

II $b \Rightarrow \exists$ a basis $\{V, U, M\} \ni C^{(5)} = C^{(4)} = 0\,,\; C^{(3)} = \text{pure imaginary}\ (\neq 0)$.

For type II *a* we have $C_{abcd}\,\overset{*}{C}{}^{abcd} = 0\,,\; C_{abcd}\,C^{abcd} > 0$; and for type II *b*: $C_{abcd}\,\overset{*}{C}{}^{abcd} = 0\,,\; C_{abcd}\,C^{abcd} < 0$. Consequently:

THEOREM 4. *If a compact Einstein-Lorentz manifold* M *is of Type II a, then* $p\,(M) = 0$ *and*

$$\int_M \left(R^{ab}\,R_{ab} - \frac{1}{3}\,r^2\right) dv > 0\,.$$

COROLLARY. *If* M *is of type IIa it does not admit a null electromagnetic field.*

THEOREM 5. *If a compact Einstein-Lorentz manifold* M *is of type IIb, then* $p\,(M) = 0$ *and*

$$\int_M \left(R^{ab}\,R_{ab} - \frac{1}{3}\,r^2\right) dv < 0\,.$$

COROLLARY. *Compact Einstein-Lorentz manifolds of type IIb admit neither perfect fluids nor electromagnetic fields (null or otherwise).*

Finally suppose M is of type II with $C_{abcd}\,C^{abcd} = 0$; then for $C^{(3)} = \alpha + i\beta$ we have $\alpha^2 - \beta^2 = 0$ and

$$C_{abcd}\,\overset{*}{C}{}^{abcd} = 24\,\alpha\beta\,.$$

Therefore at each point of M, either

$$C_{abcd}\,\overset{*}{C}{}^{abcd} = 24\,\alpha^2 > 0\,, \qquad (\alpha = \beta)\,,$$

$$\text{or} \quad C_{abcd}\,\overset{*}{C}{}^{abcd} = -\,24\,\alpha^2 < 0\,, \qquad (\alpha = -\,\beta)\,.$$

Consequently:

THEOREM 6. *If* M *is a compact Einstein-Lorentz manifold of type II (or* D*) with* $C_{abcd}\,C^{abcd} = 0$, *then* $p\,(M) \neq 0$.

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