Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Arun Verma

Hahn polynomials

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **51** (1971), n.3-4, p. 168–176.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_51_3-4_168_0>

Funzioni speciali. — *Hahn polynomials.* Nota (*) di ARUN VERMA, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano alcune caratterizzazioni dei polinomi di Hahn, si trovano alcuni sviluppi in serie e funzioni generatrici degli stessi polinomi e si discutono alcuni interessanti casi speciali.

§ 1. INTRODUCTION

In his investigations on polynomials satisfying q-difference equations Hahn [7] encountered the orthogonal polynomials $p_n(x;\beta,\gamma,\delta)$ belonging to the jump function $[\beta]_x[\gamma]_x/x!\,[\delta]_x$. These polynomials contained as special cases the polynomials of Bateman [4, 5] Krawtchouk, Charlier, Meixner, Rice [10], Jacobi, Gaganbour, Laguerre, Legendre, etc.(1) Weber and Erdélyi [12] gave the explicit representation of these polynomials as

$$p_n(x;\beta,\gamma,\delta) = \frac{[\beta]_n[\gamma]_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,-x,\beta+\gamma-\delta+n;1\\ \beta,\gamma\end{matrix}\right],$$

and a recurrence relation satisfied by them. Later on Al-Salam [1, 2] gave some elegant characterisations for these polynomials. Al-Salam [2] also studied the polynomials (henceforth referred as Hahn polynomials)

$$G_n(x;z) \equiv G_n(x;\beta,\gamma,\delta;z) = \frac{[\beta]_n[\gamma]_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,-x,\beta+\gamma-\delta+n;z\\ \beta,\gamma\end{matrix}\right],$$

which clearly reduce to the above polynomials $p_n(x;\beta,\gamma,\delta)$ for $z=1$. In this note we prove some characterisations, expansions and generating functions for Hahn polynomials and discuss some interesting special cases.

§ 2. CHARACTERISATIONS FOR $G_n(x;z)$

We begin by proving the following characterisations of the Hahn polynomials:

THEOREM I. *If* $F_n(x;\beta,\gamma,\delta;z)$ *is a polynomial of degree exactly* n *in* x; β,γ,δ,z *are independent of* x *and* $\underset{x}{\Delta} F_n(x;\beta,\gamma,\delta;z) = (\beta+\gamma-\delta+n)\,z\,F_{n-1}(x;\beta+1,\gamma+1,\delta;z)$ *and* $F_n(0;\beta,\gamma,\delta;z) = \frac{[\beta]_n[\gamma]_n}{n!}$ *then* $F_n(x;\beta,\gamma,\delta;z)$ *is nothing else than the Hahn polynomials*

$$\frac{[\beta]_n[\gamma]_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,-x,\beta+\gamma-\delta+n;z\\ \beta,\gamma\end{matrix}\right], \qquad \text{where } \underset{x}{\Delta} f(x) = f(x+1)-f(x).$$

(*) Pervenuta all'Accademia il 6 ottobre 1971.

(1) For notations and definitions see Erdélyi *et al.* [6].

Proof: Let

(1) $$F_n(x;\beta,\gamma,\delta;z)=\sum_{k=0}^{n} C_k(\beta,\gamma,\delta,n,z)\,[-x]_k,$$

where $C_k(\beta,\gamma,\delta,n,z)$ are functions of β,γ,δ,n and z and are independent of x. Then, since

$$\Delta_x F_n(x;\beta,\gamma,\delta;z)=-\sum_{k=0}^{n} kC_k(\beta,\gamma,\delta,n;z)\,[-x]_{k-1}$$

$$=-\sum_{k=0}^{n-1}(k+1)\,C_{k+1}(\beta,\gamma,\delta,n;z)\,[-x]_k$$

$$=z(\beta+\gamma-\delta+n)\,F_{n-1}(x;\beta+1;\gamma+1,\delta;z),$$

equating the coefficients of $[-x]_k$ on both the sides, we get

$$C_{k+1}(\beta,\gamma,\delta,n;z)=-\frac{(\beta+\gamma-\delta+n)}{k+1}\,zC_k(\beta+1,\gamma+1,\delta,n-1;z),$$

$$k=0,1,\cdots,n-1;\ n=1,2,3,\cdots$$

i.e.

(2) $$C_k(\beta,\gamma,\delta,n;z)=\frac{(-)^k[\beta+\gamma-\delta+n]_k}{k!}\,z^k C_0(\beta+k,\gamma+k,\delta,n-k;z).$$

Further since

$$F_n(0;\beta,\gamma,\delta;z)=\frac{[\beta]_n[\gamma]_n}{n!},$$

we have from (2.1) that

$$C_0(\beta,\gamma,\delta,n;z)=\frac{[\beta]_n[\gamma]_n}{n!}.$$

Substituting the value of $C_0(\beta,\gamma,\delta,n;z)$ in (2.2), we get, on some reduction

$$C_k(\beta,\gamma,\delta,n;z)=\frac{[\beta]_n[\gamma]_n}{n!}\,\frac{[-n]_k[\beta+\gamma-\delta+n]_k}{k!\,[\beta]_k[\gamma]_k},$$

which proves the assertion.

THEOREM II. *If* $F_n(x;\beta,\gamma,\delta;z)$ *is a polynomial of degree exactly* n *in* $(\delta+n)$ *and* x,β,γ,z *are independent of* δ *and*

$$\Delta_\delta F_n(x;\beta,\gamma,\delta;z)=-xz\,F_{n-1}(x-1;\beta+1,\gamma+1,\delta+1;z),$$

and $F_n(x;\beta,\gamma,-n;z)=\frac{[\beta]_n[\gamma]_n}{n!}$ *then* $F_n(x;\beta,\gamma,\delta;z)$ *is nothing else than the Hahn polynomials*

$$\frac{[\beta]_n[\gamma]_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,-x,\delta+n;z\\ \beta,\gamma\end{matrix}\right].$$

The proof is exactly similar to that of Theorem I, hence is omitted.

§ 3. Expansions involving Hahn polynomials

Let us consider the hypergeometric polynomial (which clearly reduces to the Hahn polynomial $G(x;z)$ for $p=0, q=2$) $f_n[x, \delta, (a_p):(b_q); z]$ defined as

$$f_n[x;z] \equiv f_n[x, \delta, (a_p):(b_q); z] = \frac{[(b_q)]_n}{n!}\, {}_{p+3}F_q\left[\begin{matrix} -n, -x, -\delta+n, (a_p); z \\ (b_q) \end{matrix}\right].$$

We can obtain without difficulty the following relations:

$$\underset{x}{\Delta^r} f_n[x, \delta, (a_p):(b_q); z] = \\ = [(a_p)]_r [-\delta+n]_r z^r f_{n-r}[x, \delta-2r, r+(a_p): r+(b_q); z],$$

$$\underset{\delta}{\Delta^r} f_n[x, \delta, (a_p):(b_q); z] = \\ = [(a_p)]_r [-x]_r z^r f_{n-r}[x-r, \delta-r, r+(a_p): r+(b_q); z],$$

$$\underset{a_j}{\Delta^r} f_n[x, \delta, (a_p):(b_q); z] = (-)^r [-x]_r [-\delta+n]_r \frac{[(a_p)]_r}{[a_j]_r} z^r \times \\ \times f_{n-r}[x-r, \delta-3r, r+(a_p): r+(b_q); z], \qquad (1 \le j \le p),$$

$$\underset{b_j}{\Delta^r} f_n[x, \delta, (a_p):(b_q); z] = (-)^{r(q-1)} \frac{[1-n-(b_q)]_r}{[1-n-b_j]_r} \times \\ \times f_{n-r}[x, \delta-r, (a_p): b_1, b_2, \cdots, b_{j-1}, b_j+r, b_{j+1}, \cdots, b_q; z], \quad (1 \le j \le q),$$

where $\Delta^{r+1} f(x) = \Delta[\Delta^r f(x)]$ and $\Delta f(x) = f(x+1) - f(x)$.

Hence in the Newton formula

$$f(x+\mu) = \sum_r \binom{\mu}{r} \underset{x}{\Delta^r} f(x),$$

using the above expressions, we get the following expansions of the hypergeometric polynomials:

$$(1) \qquad f_n[x+Y, \delta, (a_p):(b_q); z] = \sum_r \binom{Y}{r} [(a_p)]_r [-\delta+n]_r z^r \times \\ \times f_{n-r}[x, \delta-2r, r+(a_p): r+(b_q); z],$$

$$(2) \qquad f_n[x, \delta+\delta', (a_p):(b_q); z] = \sum_r \binom{\delta'}{r} [(a_p)]_r [-x]_r z^r \times \\ \times f_{n-r}[x-r, \delta-r, r+(a_p): r+(b_q); z],$$

$$(3) \qquad f_n[x, \delta, (a_{j-1}), a_j+a'_j, a_{j+1}, \cdots, a_p:(b_q); z] = \\ = \sum_r \binom{a'_j}{r} (-)^r [-x]_r [-\delta+n]_r \frac{[(a_p)]_r}{[a_j]_r} z^r \times \\ \times f_{n-r}[x-r; \delta-3r, r+(a_p): r+(b_q); z], \qquad (1 \le j \le p),$$

$$(3)' \qquad f_n[x, \delta, (a_p):(b_{j-1}), b_j + b'_j, b_{j+1}, \cdots, b_q; z] =$$
$$= \sum_r (-)^{r(q-1)} \binom{b'_j}{r} \frac{[1-(b_q)-n]_r}{[1-b_j-n]_r} \times$$
$$\times f_{n-r}[x, \delta - r, (a_p):(b_{j-1}), b_j + r, b_{j+1}, \cdots, b_q; z], \qquad (1 \leq j \leq q).$$

In the above expansions setting $p = 0, q = 2$ we get the following expansions of the Hahn polynomials (of course the formula (3.3) is meaningless for $p = 0$):

$$G_n(x+Y; \beta, \gamma, \delta; z) = \sum_r \binom{Y}{r} [\beta + \gamma - \delta + n]_r z^r G_{n-r}(x; \beta + r, \gamma + r, \delta; z),$$

$$G_n(x; \beta, \gamma, \delta + \delta'; z) = \sum_r \binom{\delta'}{r} [-x]_r z^r G_{n-r}(x - r; \beta + r, \gamma + r, \delta + r; z),$$

$$G_n(x; \beta + \beta', \gamma, -\delta + \gamma + \beta + \beta'; z) =$$
$$= \sum_r (-)^r \binom{\beta'}{r} [1 - n - \gamma]_n G_{n-r}(x; \beta + r, \gamma, \beta + \gamma - \delta; z).$$

Next, we show that if n is a positive integer

$$(4) \qquad z^n = \sum_{k=0}^{n} \frac{[(b_q)+k]_{n-k}[-n]_k}{[(a_p)]_n[-x]_n} \frac{\Gamma[1+\lambda+k]}{\Gamma[2+\lambda+n+k]} \cdot$$
$$\cdot (\lambda + 2k + 1) f_k[x, -\lambda - 1, (a_p):(b_q); z].$$

For proving (3.4) write the series definition for the polynomial $f_n(x, -\lambda - 1, (a_p):(b_q); z)$ in the right hand side and changing the order of summation to obtain

$$\frac{[(b_q)]_n}{[-x]_n[(a_p)]_n} \sum_{r=0}^{n} \frac{[-x]_r[(a_p)]_r}{r!\,[(b_q)]_r} z^r \sum_{k=r}^{n} \frac{[-n]_k[-k]_r}{k!} \Gamma[\lambda + 1 + k + r](\lambda + 2k + 1).$$

Writing $r + p$ for k and simplifying it becomes

$$(5) \quad \frac{[(b_q)]_n}{[-x]_n[(a_p)]_n} \sum_{r=0}^{n} \frac{(-)^r[-x]_r[(a_p)]_r[-n]_r}{r!\,[(b_q)]_r} \cdot (\lambda + 2r + 1) \cdot \frac{\Gamma[\lambda + 1 + 2r]}{\Gamma[\lambda + 2 + n + r]} \times$$
$$\times {}_3F_2 \left[\begin{matrix} \lambda + 2r + 1, -n + r, \frac{3}{2} + \frac{\lambda}{2} + r; 1 \\ 2 + \lambda + n + r, \frac{1}{2} + \frac{\lambda}{2} + r \end{matrix} \right].$$

Now making use of Dixon's summation formula for a well-poised ${}_3F_2(+1)$ [9; pp. 92] it is clear that the sum of the inner series in (3.5) is zero unless $r = n$ and in that case it is equal to one. Making use of it in (3.5), we get (3.4).

An alternative proof of (3.4) could be given using a series transform due to Gould [8]. Gould has shown that if we write

$$\text{(6)} \qquad G(n, a, b, f) = \sum_{k=0}^{n} (-)^k \binom{n}{k} \binom{a+n+bk}{k} f(k),$$

where $f(k)$ is independent of n and $f(0) = 1$, then

$$\text{(7)} \qquad \binom{a+bn+n}{n} f(n) = \sum_{k=0}^{n} (-)^k \frac{a+k+bk+1}{a+n+bn+1} \binom{a+n+bn+1}{n-k} G(k, a, b, f).$$

One can easily verify that the hypergeometric polynomial

$$f_n[x, \delta, (a_p) : (b_q); z]$$

could alternatively be written as

$$\text{(8)} \qquad \frac{n!}{[(b_q)]_n} \binom{\lambda+n}{n} f_n[x, -\lambda-1, (a_p) : (b_q); z] =$$

$$= \sum_{k=0}^{n} (-)^k \binom{n}{k} \binom{\lambda+n+k}{n} \binom{\lambda+k}{k} \frac{k!\,[-x]_k\,[(a_p)]_k\, z^k}{[(b_q)]_k}.$$

Hence taking

$$f(k) = \binom{\lambda+k}{k} \frac{k!\,[-x]_k\,[(a_p)]_k\, z^k}{[(b_q)]_k}$$

and

$$G(n, \lambda, 1, f) = \frac{n!}{[(b_q)]_n} \binom{\lambda+n}{n} f_n[x, -\lambda-1, (a_p) : (b_q); z],$$

and using (3.6), we get (3.4).

(3.4) for $p = 0, q = 2$ gives the following result for the Hahn polynomials:

$$z^n = \sum_{k=0}^{n} \frac{[\beta+k]_{n-k}\,[\gamma+k]_{n-k}\,[-n]_k}{[-x]_n} \cdot \frac{\Gamma[1+\lambda+k]}{\Gamma[2+\lambda+n+k]} (\lambda+2k+1) \times$$

$$\times G_k(x; \beta, \gamma, \beta+\gamma-\lambda-1; z),$$

which in turn for $x = -\gamma$ reduces to a known result for Jacobi polynomials [3].

Next, we prove that if n is a positive integer, then

$$\text{(9)} \qquad (y+z)^n = \frac{n!\,(-)^{nq}}{[(a_p)]_n\,[-x]_n} \sum_{k=0}^{n} (-)^{k(q+1)} \frac{\Gamma[1+\lambda+k]}{\Gamma[2+\lambda+n+k]} (\lambda+2k+1) \times$$

$$\times f_k[x, -\lambda-1, (a_p) : (b_q); z] \times$$

$$\times f_{n-k}[n-x-1, 1+\lambda+2n, 1-(a_p)-n : 1-(b_q)-n; (-)^{q-p-1} y].$$

To prove (3.9)

$$(y+z)^n = \sum_{r=0}^{n} \binom{n}{r} y^{n-r} z^r.$$

Applying (3.4) we find

$$(y+z)^n = \sum_{r=0}^{n} \binom{n}{r} y^{n-r} \sum_{k=0}^{r} \frac{[(b_q)+k]_{r-k}[-x]_k \Gamma[1+\lambda+k]}{[(a_p)]_r [-x]_r \Gamma[2+\lambda+k+r]} \cdot$$

$$\cdot (\lambda+2k+1) f_k[x, -\lambda-1, (a_p):(b_q); z].$$

Changing the order of summation and inverting the resulting inner series we get (3.9). Clearly (3.9) reduces for $p=0$, $q=2$ to the following result for the Hahn polynomials

$$(y+z)^n = \frac{n!}{[-x]_n} \sum_{k=0}^{n} (-)^k \frac{\Gamma[1+\lambda+k]}{\Gamma[2+\lambda+k+n]} (\lambda+2k+1) G_k(x;\beta,\gamma,\beta+\gamma-\lambda-1;z) \times$$

$$\times G_{n-k}(n-x-1, 1-\beta-n, 1-\gamma-n, 3-\beta-\gamma+\lambda; -y),$$

which for $x=-\gamma$ reduces to a known result for Jacobi polynomials [3]. Whereas (3.9) reduces to (3.4) for $y=0$.

§ 4. GENERATING FUNCTIONS

We begin by observing that using the definition (3.8) for Hahn polynomials (i.e. the case when $p=0$, $q=2$) and the following series transform (Gould [8])

$$\sum_{k=0}^{n} \binom{a+k+bk}{k} z^k f(k) = v^{a+1} \sum_{n=0}^{\infty} (-)^n G_n(n, a, b, f)(v-1)^n,$$

where $z = \frac{v-1}{v^{b+1}}$ and $f(k)$ and $G(n, a, b, f)$ are defined by (3.6), one could verify in a straight forward the following known generating function for the Hahn polynomials (Al-Salam [2]):

(1)
$$\sum_{n=0}^{\infty} \frac{(-)^n [1+\lambda]_n}{[\beta]_n [\gamma]_n} G_n(x;\beta,\gamma,\beta+\gamma-\lambda-1;z) t^n =$$

$$= {}_3F_2\left[\begin{matrix} -x, \frac{1+\lambda}{2}, \frac{2+\lambda}{2}; \frac{4zt}{(1+t)^2} \\ \beta, \gamma \end{matrix}\right],$$

in which replacing x by $-\gamma$ we get a known generating function for the Jacobi polynomials [9; pp. 261].

Next, we prove

(2)
$$\sum_{n=0}^{\infty} \frac{\beta}{[\gamma]_n (\beta+2n)} G_n(x; 1+\beta+n, \gamma, 1+\delta+2n; z) t^n =$$

$$= \left(\frac{2}{1+\sqrt{1-4t}}\right)^{\beta} {}_3F_2\left[\begin{matrix} -x, \frac{\beta}{2}, \beta+\gamma-\delta; \frac{-4zt}{(1+\sqrt{1-4t})^2} \\ \gamma, 1+\frac{1}{2}\beta \end{matrix}\right],$$

(3)
$$\sum_{n=0}^{\infty} \frac{(-t)^n}{[\gamma]_n} G_n(x\,;\beta+\lambda n,\gamma,\delta+\lambda n\,;z) =$$

$$= \frac{(1+v)^{-\beta}}{1+(\lambda+1)v}\ {}_2F_1\left[\begin{matrix}-x,\beta+\gamma-\delta\,;\dfrac{zt}{(1+v)^{\lambda+1}}\\ \gamma\end{matrix}\right],$$

where $v(1+v)^{\lambda} = t$.

For proving (4.2) we write the series definition for $G_n(x\,;z)$ in the left hand side of (4.2) and change the order of summation to get

$$\sum_{r=0}^{\infty} \frac{[-x]_r\,[\beta+\gamma-\delta]_r\,[\beta]_{2r}}{[1]_r\,[\gamma]_r\,[1+\beta]_{2r}}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}\beta+\frac{1}{2}+r,\frac{1}{2}\beta+r\,;4t\\ 1+\beta+2r\end{matrix}\right](-zt)^r.$$

On summing the inner ${}_2F_1$ by making use of the known formula [9; pp. 70, Ex. 10]

(4)
$${}_2F_1\left[\begin{matrix}\gamma,\gamma-\frac{1}{2}\,;v\\ 2\gamma\end{matrix}\right] = \left(\frac{2}{1+\sqrt{1-v}}\right)^{2\gamma-1},$$

we get (4.2). The proof of (4.3) runs on exactly similar lines but in this case instead of using (4.4) use is made of the following known formula

(5)
$$\sum_{n=0}^{\infty} \frac{[-a-bn-n]_n}{n!}(-t)^n = \frac{(1+v)^{a+1}}{1-bv},$$

where $t = v/(1+v)^{b+1}$. The generating function (4.3) could have also been deduced as a special case of a known result of Verma [11].

Next, we consider the following generating function of Hahn polynomials due to Al–Salam [2]:

$$\sum_{n=0}^{\infty} \frac{[A]_n}{[\beta]_n\,[\gamma]_n} G_n(x\,;\beta,\gamma,\delta+n\,;z)\,t^n = (1-t)^{-A}\ {}_3F_2\left[\begin{matrix}-x,\beta+\gamma-\delta,A\,;\dfrac{-zt}{1-t}\\ \beta,\gamma\end{matrix}\right].$$

In this result writing $A=\beta$, $\delta=\gamma/c$, $z=1$ and letting $\gamma\to\infty$, we get the following known generating function for Meixner polynomials [6; 10.24 (13)]

(6)
$$\sum_{n=0}^{\infty} m_n(x\,;\beta,c)\,t^n = \left(1-\frac{z}{c}\right)^x (1-z)^{-\beta-x},$$

where the Meixner's polynomials $m_n(x\,;\beta,c)$ are defined as

$$m_n(x\,;\beta,c) = [\beta]_n\ {}_2F_1\left[\begin{matrix}-n,-x\,;1-\dfrac{1}{c}\\ \beta\end{matrix}\right].$$

Using (4.6), we can write

$$(7)\quad \left[\sum_{n_1=0}^{\infty} m_{n_1}(x_1 ; \beta_1 , c) \frac{z^{n_1}}{n_1!}\right]\left[\sum_{n_2=0}^{\infty} m_{n_2}(x_2 ; \beta_2 , c) \frac{z^{n_2}}{n_2!}\right]\cdots\left[\sum_{n_r=0}^{\infty} m_{n_r}(x_r ; \beta_r , c) \frac{z^{n_r}}{n_r!}\right]=$$

$$=\left(1-\frac{z}{c}\right)^{x_1+x_2+\cdots+x_r}(1-z)^{-(x_1+x_2+\cdots+x_r)-(\beta_1+\beta_2+\cdots+\beta_r)}=$$

$$=\sum_{n=0}^{\infty} m_n(x_1+x_2+\cdots+x_r ; \beta_1+\beta_2+\cdots+\beta_r ; c)\frac{z^n}{n!}$$

using (4.6). Equating the coefficients of z^n on both sides, we get

$$(8)\quad \sum_{n_1+n_2+\cdots+n_r=n} m_{n_1}(x_1 ; \beta_1 , c)\, m_{n_2}(x_2 ; \beta_2 , c)\cdots m_{n_r}(x_r ; \beta_r , c)\cdot$$

$$\cdot\frac{1}{n_1!\, n_2!\cdots n_r!}=\frac{1}{n!}\, m_n(x_1+x_2+\cdots+x_r ; \beta_1+\beta_2+\cdots+\beta_r , c).$$

Next, we could have written (4.7) as

$$\left[\sum_{n_1=0}^{\infty} m_{n_1}(x_1 ; \beta_1 , c)\frac{z^{n_1}}{n_1!}\right]\cdots\left[\sum_{n_r=0}^{\infty} m_{n_r}(x_r ; \beta_r , c)\frac{z^{n_r}}{n_r!}\right]=$$

$$=(1-z)^{-(\beta_2+\cdots+\beta_r)}\left(1-\frac{z}{c}\right)^{x_1+x_2+\cdots+x_r}(1-z)^{-(x_1+\cdots+x_r)-\beta_1}=$$

$$=(1-z)^{-(\beta_2+\cdots+\beta_r)}\sum_{n=0}^{\infty} m_n(x_1+x_2+\cdots+x_r ; \beta_1 , c)\frac{z^n}{n!}\cdot$$

We have the following interesting relation involving Meixner polynomials on equating the coefficients of z^n on both sides

$$(9)\quad \sum_{n_1+n_2+\cdots+n_r=n} m_{n_1}(x_1 ; \beta_1 , c)\, m_{n_2}(x_2 ; \beta_2 , c)\cdots m_{n_r}(x_r ; \beta_r , c)\cdot$$

$$\cdot\frac{1}{n_1!\,n_2!\cdots n_r!}=\sum_{p=0}^{n}\frac{[\beta_2+\cdots+\beta_r]_{n-p}}{p!\,(n-p)!}\, m_p(x_1+\cdots+x_r ; \beta_1 , c).$$

(4.8) and (4.9) yield the following addition formula for Meixner polynomials

$$\sum_{p=0}^{n}\frac{[\beta_2+\cdots+\beta_r]_{n-p}}{p!\,(n-p)!}\, m_p(x_1+x_2+\cdots+x_r ; \beta_1 , c)=$$

$$=\frac{1}{n!}\, m_n(x_1+x_2+\cdots+x_r ; \beta_1+\beta_2+\cdots+\beta_r , c).$$

Lastly, making use of (4.6) twice, we get

$$(10)\quad \left[\sum_{n=0}^{\infty} m_n(x ; \beta , c)\frac{z^n}{n!}\right]\left[\sum_{r=0}^{\infty} m_r(x ; \beta , c)\frac{(-z)^r}{r!}\right]=\left(1-\frac{z^2}{c^2}\right)(1-z^2)^{-x-\beta}=$$

$$=\sum_{p=0}^{\infty} m_p(x ; \beta , c^2)\frac{z^{2p}}{p!}$$

which yields on equating the coefficients of z^p in both sides

$$\sum_{n=0}^{p}(-)^{p-n}\binom{p}{n} m_n(x ; \beta , c)\, m_{p-n}(x ; \beta , c)=\begin{cases}0 & \text{when } p \text{ is odd}\\ m_p(x ; \beta , c^2) & \text{when } p \text{ is even.}\end{cases}$$

REFERENCES

[1] AL-SALAM N. A., *Orthogonal polynomials of hypergeometric type*. « Duke Math. Jour. », *33*, 109–122 (1966).

[2] AL-SALAM N. A., *A class of hypergeometric polynomials*. « Annali Mat. Pura Appl. », *75*, 95–120 (1967).

[3] AL-SALAM W. A., *Some relations involving Jacobi polynomials*. « Portugaliae Math. », *15*, 73–77 (1956).

[4] BATEMAN H., *Some properties of a certain set of polynomials*. « Tohoku Math. Jour. », *37*, 23–28 (1933).

[5] BATEMAN H., *The polynomials* $F_n(x)$. « Annales of Math. », *35*, 767–775 (1944).

[6] ERDÉLYI A. *et al.*, *Higher trancendental functions*. Vol. II. McGraw Hill (1955).

[7] HAHN W., *Uber orthogonal Polynome, die q–Differenzengecic–hungen genugen*. « Math. Nacher. », *2*, 4–34 (1949).

[8] GOULD H. W., *A new series transform with applications to Bessel, Legendre, Tchebycheff polynomials*. « Duke Math. Jour. », *31*, 325–334 (1964).

[9] RAINVILLE E. D., *Special functions*. MacMillan Co., New York 1960.

[10] RICE S. O., *Some properties of* ${}_3F_2(-n, -n-1, 1; p, v)$. « Duke Math. Jour. », *6*, 108–119 (1940).

[11] VERMA A., *A class of expansions of G-functions and the Laplace transform*. « Math. Comp. », *19*, 664–666 (1965).

[12] WEBER M. and ERDÉLYI A., *On the finite difference analogue of Rodrigue's formula*. « Amer. Math. Monthly », *59*, 163–168 (1952).