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On the theory of fixed points for some classes of mappings. Nota V

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Equazioni funzionali. — *On the theory of fixed points for some classes of mappings.* Nota V (*) di VASILE ISTRĂŢESCU e ANA ISTRĂŢESCU, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia (X, d) è uno spazio metrico completo e $T : X \to X$ una trasformazione tale che $d(Tx, Ty) \leq kd(x, y)$ dove $k \in (0, 1)$ e $x, y \in X$, allora per il teorema di Picard–Banach T ha un unico punto fisso.

Negli ultimi anni varie generalizzazioni di questo risultato sono state ottenute.

Principale scopo di questa Nota è di ottenere nuove generalizzazioni dei risultati di questo tipo.

Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq kd(x, y)$$

where $k \in (0, 1)$ and $x, y \in X$, then by Picard-Banach Theorem T has a unique fixed point. In recent years many generalizations of this result were derived. Our aim in this Note is to obtain new generalizations of results of this type.

1. Let (X, d), as above be a complete metric space and $T : X \to X$ be a mapping of X into itself. In [7], a sufficient condition for the existence of fixed points was given. We give here a generalization of this result. Our generalization is inspired from [1], [7], [8].

We consider mappings $T : X \to X$ which satisfy the following condition

$$(*) \qquad d(Tx, Ty) \leq \frac{1}{2}\left[\psi(d(x, Tx)) + \psi(d(y, Tx))\right]$$

where ψ is some function defined on the closure of the range of d.

The case considered in [7] is for the function $\psi(t) = \alpha t$, $\alpha < 1$. Our results are as follows.

THEOREM 1. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *satisfy the condition* $(*)$ *where* $\psi : \overline{P} - \{0\}$, *where* P *is the range of* d $(P = \{d(x, y), x, y \in X\})$ *and is upper semicontinuous from the right on* $\overline{P}$ *and satisfy* $\psi(t) < t$ *for all* $t \in \overline{P} - \{0\}$. *Then* T *has a unique fixed point* x_0 *and for each* $x \in X$, $T^n x \to x_0$.

Proof. Given $x \in X$ we consider the sequence

$$c_n(x) = c_n = d(T^n x, T^{n-1} x)$$

(*) Pervenuta all'Accademia il 3 settembre 1971.

and we prove that $c_n \to 0$. First we show that $\{c_n\}$ is decreasing, for this we remark that

$$d(T^n x, T^{n-1} x) \leq \frac{1}{2} [\psi(d(T^{n-1} x, T^n x)) + \psi(d(T^{n-2} x, T^{n-1} x))] \leq$$

$$\leq \frac{1}{2} (d(T^n x, T^{n-1} x)) + \frac{1}{2} (d(T^{n-2} x, T^{n-1} x)).$$

Which gives

$$\frac{1}{2} d(T^{n-1} x, T^n x) \leq \frac{1}{2} d(T^{n-2} x, T^{n-1} x)$$

and our assertion is proved.

Let $c = \lim c_k$. If $c > 0$ we have

$$c_{n+1} < \frac{1}{2} [\psi(c_{n+1}) + \psi(c_n)]$$

so that

$$c \leq \limsup_{t \to c^+} \psi(t) \leq \psi(c)$$

which is a contradiction and thus $c = 0$. We show that $\{T^n x\}$ is a Cauchy sequence.

As in [1] we have (with notations of [1])

$$d_k \leq c_k + \varepsilon .$$

But since

$$d_k = d(T^m x, T^n x) \leq 2 c_k + d(T^{m+1} x, T^{n+1} x) \leq$$

$$\leq 2 c_k + \frac{1}{2} [\psi(d(T^{n+1} x, T^n x)) + \psi(d(T^{m+1} x, T^m x))] =$$

$$= 2 c + \frac{1}{2} [\psi(c_n) + \psi(c_{m+1})]$$

and for $k \to \infty$,

$$\varepsilon \leq \psi(\varepsilon)$$

which is a contradiction. It is clear that

$$x_0 = \lim T^n x$$

is a fixed point for T. From (*) it is clear that x_0 is the unique fixed point for T. The Theorem is proved.

2. In [10] the Theorem of Picard-Banach was generalized to a new class of mappings. This suggests the following introduction of a class of mappings as follows: for α, β such that

$$\alpha \leq d(x, Tx), \; d(y, Ty) \leq \beta$$

there exists $q(\alpha, \beta) < \frac{1}{2}$ such that

$$d(\mathrm{T}x, \mathrm{T}y) \leq q(\alpha, \beta)\,[d(x, \mathrm{T}x) + d(y, \mathrm{T}y)]\,.$$

Then the sequence $\{x_n\}_0^\infty$ where $x_{n+1} = \mathrm{T}x_n$, converges and $x_0 \in \mathrm{X}$ is given.

Proof. We consider the sequence of numbers

$$c_n = d(x_{n+1}, x_n) = d(\mathrm{T}x_n, \mathrm{T}x_{n-1})$$

and we show that $c_n \to 0$.

First we remark that $\{c_n\}$ is decreasing. Indeed,

$$c_n \leq q(n, n-1)\,[d(x_n, \mathrm{T}x_n) + d(x_{n-1}, \mathrm{T}x_{n-1})] = \\ = q(n, n-1)\,[c_n + c_{n-1}]$$

which gives

$$(*) \qquad c_n \leq \frac{q(n, n-1)}{1 - q(n, n-1)}\, c_{n-1}$$

where $q(n, n-1)$ corresponds to $\alpha = \min\{d(x_n, \mathrm{T}x_n), d(x_{n-1}, \mathrm{T}x_{n-1})\}$ and $\beta = \max\{d(x_n, \mathrm{T}x_n), d(x_{n-1}, \mathrm{T}x_{n-1})\}$.

Let $\alpha^* = \lim c_n$. If $\alpha^* > 0$ we have,

$$c_{n+m} \leq q(\alpha^*, \alpha^* + 1)^m (\alpha^* + 1)$$

which gives a contradiction. Thus $\alpha^* = 0$.

We show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{\mathrm{T}^n x_0\}$ is not a Cauchy sequence and thus there is an $\varepsilon > 0$ and sequences of integers $m(k), n(k)$ with

$$(1) \qquad m(k) > n(k) \geq k$$

$$(2) \qquad d_k = d(\mathrm{T}^m x_0, \mathrm{T}^n x_0) \geq \varepsilon$$

and we may assume that

$$d(\mathrm{T}^{m-1} x_0, \mathrm{T}^n x_0) \leq \varepsilon\,.$$

But

$$d_k \leq d(\mathrm{T}^m x_0, \mathrm{T}^{m-1} x_0) + d(\mathrm{T}^{m-1} x_0, \mathrm{T}^n x_0) \leq d_k + \varepsilon$$

which gives $\lim\limits_{k\to\infty} d_k = 0$.

Since

$$d_k = d(\mathrm{T}^m x_0, \mathrm{T}^n x_0) \leq d(\mathrm{T}^m x_0, \mathrm{T}^{m+1} x_0) + d(\mathrm{T}^{m+1} x_0, \mathrm{T}^{n+1} x_0) + \\ + d(\mathrm{T}^n x_0, \mathrm{T}^{n+1} x_0) \leq 2c_k + q(\varepsilon, \varepsilon + 1)\, d_k$$

and for $k \to \infty$ we obtain a contradiction. Thus $\{\mathrm{T}^n x_0\}$ is a Cauchy sequence and it is clear that

$$\xi = \lim \mathrm{T}^n x_0$$

is a unique fixed point for T. The assertion is proved.

3. In [8] a Theorem on fixed points of Krasnoselskii type was given. Our aim in this section is to prove a Theorem of the same type.

For this we need notion of Kuratovski number $\alpha(\cdot)$ of a bounded set in a metric space (X, d). For a bounded set $A \subset X$, $\alpha(A)$ denote the infimum of all numbers $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ε.

For properties of $\alpha(\cdot)$ see [3], [6].

DEFINITION 3.1 [5]. *A continuous mapping* $T : X \to X$ *is called locally power densifying if for each bounded set* $A \subset X$ *there exists an integer* $n = n(A)$ *such that*

$$\alpha(T^n A) < \alpha(A) .$$

For $n(A) = 1$ this class was considered by many Authors and T is called in this case densifying [3].

If

$$\alpha(T^n A) \leq k\alpha(A) \qquad k \in (0, 1)$$

and k is independent of A we call such mappings locally power α–contractions.

DEFINITION 3.2. *Let* F *be a real lower semicontinuous function on* $X \times X$ *and* $T : X \to X$. *The mapping* T *is called locally power weakly* F *contractive if for each* $x \in X$ *there exists an integer* $n = n(x)$ *such that for* $x, y \in X$, $x \neq y$

$$F(T^n x, T^n y) < F(x, y) .$$

If $F = d$ (the distance) we call locally power weakly contractive. Our result is the following.

THEOREM. *Let* S *be a bounded convex set in a Banach space* X *and* $T = A + B$ *where* $Ax + By \in S$ *whenever* $x, y \in S$. *Suppose now that*

1) A *is linear and locally power weakly* F–*contractive, and locally power densifying*;

2) B *is completely continuous*

then $Tx = x$ *has a solution in* S.

Proof. We define for each $y \in S$ the mapping

$$\tilde{T}x = Ax + By$$

which has the property that it is locally power densifying and locally power weakly contractive. From this we have that there exists a unique point $x \in S$ such that $\tilde{T}x = x$.

Indeed we consider an arbitrary element $x_0 \in S$ and

$$A = \{\tilde{T}^n x_0\}$$

which is a bounded set in X. Clearly $\tilde{T}A \subset A$. We show that A is relatively compact. Suppose that this is not true. For $n = n(A)$ we have

$$A = \tilde{T}^n A \cup \{x_0, \tilde{T}x_0 \cdots \tilde{T}^{n-1} x_0\}$$

and thus

$$\alpha(A) = \alpha(T^n A) < \alpha(A)$$

which is a contradiction. We consider now, on the compact set $\bar{A}$ the function

$$\psi : \bar{A} \to R$$

defined as follows: $\varphi(x) = F(x, Tx)$ and thus since it is lower semicontinuous (as a restriction of a lower semicontinuous function), it has a minimum point $\xi \in \bar{A}$. We suppose now that $\xi \neq T\xi$ and let $n = n(\xi)$. We have

$$\varphi(T^n(\xi)) = F(T^n\xi, T^{n+1}\xi) < F(\xi, T\xi)$$

i.e. ξ is not a minimum point. Clearly ξ is the unique fixed point.

The proof of the Theorem may be continued in this way: define the operator $L : X \to X$ in this way

$$Ly = ALy + By \qquad y \in S .$$

From property 1 of A we have that $(I - A)^{-1}$ exists and thus

$$Ly = (I - A)^{-1} By .$$

From Schauder fixed point Theorem, there exists y_0 such that

$$Ly_0 = y_0$$

This gives that $Tx = x$ has a solution.

REMARK. 1) It is easy to see that the Theorem holds in more general conditions; for example it is sufficient that B be α–contraction in the sense of G. Darbo with the constant $k < \dfrac{1}{\|(I - A)^{-1}\|}$.

2) The case of contractions is the Theorem 2 of [8].

3) Another possible extension of results of Dubrowskii [2], Granas [4] Zarantonello [9] and Nashed and Wong [8] using the Kuratowski number will be given in [6].

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