Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

S. C. Srivastava, R. S. Sinha

Asymptotic lines in a hypersurface of a Finsler space

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **51** (1971), n.1-2, p. 44–52.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_51_1-2_44_0>

Geometria differenziale. — *Asymptotic lines in a hypersurface of a Finsler space.* Nota (*) di S. C. SRIVASTAVA e R. S. SINHA, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Hayden [1] ha studiato le linee asintotiche di ordine p in un sottospazio di uno spazio riemanniano ed ha ottenuto una espressione per la curvatura k_m in qualsiasi punto di una linea asintotica di massimo ordine. In questo lavoro gli Autori hanno voluto studiare le medesime proprietà per una ipersuperficie di uno spazio di Finsler.

1. INTRODUCTION.

Let us consider a Finsler space F_n of n dimensions referred to a local coordinate system x^i (henceforth the Latin indices i, j, k vary from 1 to n), whose metric function $F(x^i, x'^i)$ satisfies the conditions usually imposed upon such a function ([2] Ch. I.). The metric tensor of F_n is defined by

$$g_{ij}(x, x') = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, x') \quad (2)$$

and since $F(x^k, x'^k)$ is positively homogeneous of degree one in x'^k, the tensor $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}(x, x')$ satisfies the identities

$$C_{ijk}(x, x')\, x'^i = C_{ijk}(x, x')\, x'^j = C_{ijk}(x, x')\, x'^k = 0 .$$

A hypersurface F_{n-1} of F_n may be represented parametrically by the equations

$$x^i = x^i(u^\alpha) \qquad (i = 1, \cdots, n ; \alpha = 1, \cdots, n-1) \tag{1.1}$$

where u^α denote the parameters of the hypersurface F_{n-1} (henceforth Greek indices vary from 1 to $n - 1$). It will be assumed throughout that the functions (1.1) are of class C^3 and the matrix with the entries $B^i_\alpha = \partial_\alpha x^i$ has rank $n - 1$. For the sake of brevity we shall use the following notations:

$$B^i_{\alpha\beta} = \partial_\alpha \partial_\beta x^i \quad , \quad B^{ij\cdots k}_{\alpha\beta\cdots\gamma} = B^i_\alpha B^j_\beta \cdots B^k_\gamma .$$

A hypersurface vector u'^α possesses components x'^i with respect to the coordinate system of F_n, which are related by

$$x'^i = B^i_\alpha u'^\alpha. \tag{1.2}$$

(*) Pervenuta all'Accademia l'11 agosto 1971.

(1) The numbers in the square brackets refer to the references given in the end.

(2) $\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial x'^i$ and $x'^i = \dfrac{dx^i}{ds}$, where s is the arc length.

The metric tensor g_{ij} induces a metric tensor $g_{\alpha\beta}$ on F_{n-1} by means of the equation

$$g_{\alpha\beta}(u^{\gamma}, u'^{\gamma}) = g_{ij}(x^{k}, x'^{k})\, B^{ij}_{\alpha\beta}. \tag{1.3}$$

A unit normal N^{i} to the hypersurface is defined uniquely at each point of F_{n-1} with respect to a direction x'^{k} tangential to F_{n-1} by means of the following relations:

$$g_{ij}(x, x')\, N^{i} N^{j} = 1 \tag{1.4}$$

and

$$N_{i}\, B^{i}_{\alpha} = 0 \tag{1.5}$$

where

$$N_{i} = g_{ij} N^{j}.$$

The intrinsic connection parameters of F_{n-1} are given by [3]

$$\hat{\Gamma}^{\varepsilon}_{\alpha\beta} = \gamma^{\varepsilon}_{\alpha\beta} - (C^{\varepsilon}_{\sigma\beta}\, G^{\sigma}_{\alpha} + C^{\varepsilon}_{\sigma\alpha}\, G^{\sigma}_{\beta} - g^{\varepsilon\sigma}\, C_{\alpha\beta\nu}\, G^{\nu}_{\sigma}).$$

With the help of this connection the mixed intrinsic covariant derivative is defined by

$$T^{i}_{\alpha|\gamma} = \partial_{\gamma} T^{i}_{\alpha} - \dot{\partial}_{\lambda} T^{i}_{\alpha}\, \hat{\Gamma}^{\lambda}_{\varepsilon\gamma}\, u'^{\varepsilon} - \hat{\Gamma}^{\beta}_{\alpha\gamma}\, T^{i}_{\beta} + \Gamma^{*i}_{hk}\, T^{h}_{\alpha}\, B^{k}_{\gamma}$$

where Γ^{*i}_{hk} are connection coefficients for F_{n} and satisfy the relation

$$\dot{\partial}_{l}\, \Gamma^{*i}_{hk}\, x'^{h}\, x'^{k} = 0. \tag{1.6}$$

We can also define the following mixed tensor with the help of this connection

$$J^{i}_{\alpha\beta} = B^{i}_{\alpha/\beta} = N^{i}\, \tilde{\Omega}_{\alpha\beta} - B^{i}_{\varepsilon}\, \Lambda^{\varepsilon}_{\alpha\beta} \tag{1.7}$$

where $\tilde{\Omega}_{\alpha\beta}$ are to be considered as the coefficients of the second fundamental form of F_{n-1} [2] and $\Lambda^{\varepsilon}_{\alpha\beta}$ is given by

$$\Lambda^{\varepsilon}_{\alpha\beta} = g^{\varepsilon\gamma}\, \Lambda_{\alpha\gamma\beta} \tag{1.8}$$

where

$$\Lambda_{\alpha\gamma\beta} = (M_{\beta\gamma}\, \tilde{\Omega}_{\alpha\sigma} + M_{\alpha\gamma}\, \tilde{\Omega}_{\beta\sigma} - M_{\alpha\beta}\, \tilde{\Omega}_{\gamma\sigma})\, u'^{\sigma} -$$
$$- (M_{\lambda\sigma}\, C^{\lambda}_{\beta\gamma} + M_{\lambda\beta}\, C^{\lambda}_{\alpha\gamma} - M_{\lambda\gamma}\, C^{\lambda}_{\beta\alpha})\, N.$$

In the above relation $\tilde{\Omega}_{\sigma\lambda}\, u'^{\sigma} u'^{\lambda} = N(u, u')$ and the tensor $M_{\alpha\beta}$ is defined by $M_{\alpha\beta} = M_{ij}\, B^{ij}_{\alpha\beta}$ where $M_{ij} = C_{ijk}\, N^{k}$. The quantities $\Lambda^{\varepsilon}_{\alpha\beta}$ satisfy the following relations

$$\Lambda^{\varepsilon}_{\alpha\beta}\, u'^{\alpha} = N M^{\varepsilon}_{\beta} \tag{1.9}$$

and

$$\Lambda^{\varepsilon}_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} = 0 . \tag{1.10}$$

Let the curvatures and unit normals relative to F_{n-1} of a curve in F_n be k'_r $(r = 1, 2, 3, \cdots, n-2)$ and η^{α}_r $(r = 2, 3, \cdots, n-1)$ and those relative to F_n be k_r $(r = 1, 2, 3, \cdots, n-1)$ and ξ^i_r $(r = 2, 3, \cdots n)$. The unit tangents will be denoted by η^{α}_1 and ξ^i_1 according as we regard it as a vector in F_{n-1} or in F_n. It is clear from (1.2) that

$$\xi^i_1 = \eta^{\alpha}_1\, B^i_{\alpha} \tag{1.11}$$

all along every curve in F_n; but in general

$$\xi^i_r \neq \eta^{\alpha}_r\, B^i_{\alpha} \qquad \text{for} \quad r > 1 .$$

2. Asymptotic line of order p

Definition. A curve in a hypersurface F_{n-1} of a Finsler space F_n is defined as an *asymptotic line of order* p of F_{n-1} if at every point of the curve its first p normals relative to F_n are all tangential to F_{n-1}, where p is essentially less than $n-1$. So we must have

$$\xi^i_{r+1}\, N_i = 0 \qquad (r = 1, 2, 3, \cdots, p). \tag{2.1}$$

Now we will obtain a set of conditions for an asymptotic line of order p.

Theorem (2.1). – *The necessary and sufficient condition for a curve in* F_{n-1} *to be an asymptotic line of order* p *is*

$$\xi^i_{r+1} = \eta^{\alpha}_{r+1}\, B^i_{\alpha} \qquad (r = 1, 2, 3, \cdots, p) \tag{2.2}$$

or

$$k_r = k'_r \qquad (r = 1, 2, \cdots, p). \tag{2.3}$$

Proof. We will prove it in three parts. In first part it will be proved that the conditions are necessary, by the method of induction. In next two parts we will show that the conditions are sufficient.

(i) Suppose (2.2) and (2.3) are satisfied along an asymptotic line of order p, then we shall prove that the similar results are also satisfied along an asymptotic line of order $p+1$.

Since an asymptotic line of order $p+1$ is, in fact, an asymptotic line of every lower order, we have from (2.2),

$$\xi^i_{p+1} = \eta^{\alpha}_{p+1}\, B^i_{\alpha} .$$

Differentiating this along the curve we have

$$\frac{\delta}{\delta s}\, \xi^i_{p+1} = B^i_{\alpha}\, \frac{\delta}{\delta s}\, \eta^{\alpha}_{p+1} + \eta^{\alpha}_{p+1}\, \frac{\delta}{\delta s}\, B_{\alpha}$$

where $\frac{\delta}{\delta s}$ is the intrinsic derivative along the curve. With the help of (1.7) and the Frenet formula

$$\frac{\delta}{\delta s}\,\xi^i_{p+1} = k_{p+1}\,\xi^i_{p+2} - k_p\,\xi^i_p \quad , \qquad k_0 = k_n = 0$$

we have

$$k_{p+1}\,\xi^i_{p+2} - k_p\,\xi^i_p = B^i_\alpha\,(k'_{p+1}\,\eta^\alpha_{p+2} - k'_p\,\eta^\alpha_p) + J^i_{\alpha\beta}\,\eta^\alpha_{p+1}\,\eta^\beta_1 ,$$

which, by virtue of (2.2) and (2.3), reduces to

$$k_{p+1}\,\xi^i_{p+2} = B^i_\alpha\,k'_{p+1}\,\eta^\alpha_{p+2} + J^i_{\alpha\beta}\,\eta^\alpha_{p+1}\,\eta^\beta_1 . \tag{2.4}$$

Multiplying (2.4) by N_i and using (2.1), which also defines an asymptotic line of order $p+1$, we have

$$J^i_{\alpha\beta}\,N_i\,\eta^\alpha_{p+1}\,\eta^\beta_1 = 0 .$$

From (1.5) and (1.7) we have

$$\tilde{\Omega}_{\alpha\beta}\,\eta^\alpha_{p+1}\,\eta^\beta_1 = 0 . \tag{2.5}$$

On substituting (1.7), (1.9) and (2.5) in (2.4), we have

$$k_{p+1}\,\xi^i_{p+2} = k'_{p+1}\,\eta^\alpha_{p+2}\,B^i_\alpha - B^i_\varepsilon\,N M^\varepsilon_\alpha\,\eta^\alpha_{p+1} .$$

As the curve is an asymptotic line of first order also, for which it is known that

$$\tilde{\Omega}_{\alpha\beta}\,\eta^\alpha_1\,\eta^\beta_1 = 0 \qquad \text{i.e.} \quad N = 0 . \tag{2.6}$$

The above relation is also equivalent to

$$J^i_{\alpha\beta}\,\eta^\alpha_1\,\eta^\beta_1 = 0 . \tag{2.7}$$

Consequently, we have

$$k_{p+1}\,\xi^i_{p+2} = k'_{p+1}\,\eta^\alpha_{p+2}\,B_\alpha$$

which implies that

$$k_{p+1} = k'_{p+1} \qquad \text{and} \quad \xi^i_{p+2} = \eta^\alpha_{p+2}\,B^i_\alpha .$$

This shows that (2.2) and (2.3) is true for $r = p+1$, that is, for asymptotic line of order $p+1$.

Now along any curve in F_{n-1}, $\xi^i_1 = \eta^\alpha_1\,B^i_\alpha$, proceeding from this as above, we have

$$k_1\,\xi^i_2 = k'_1\,\eta^\alpha_2\,B^i_\alpha + J^i_{\alpha\beta}\,\eta^\alpha_1\,\eta^\beta_1 \tag{2.8}$$

which on multiplication by N_i and using (2.1) for $r = 1$, we have (2.6). So on substituting from (1.10), (2.6) in (2.8), we obtain a relation, which implies that $k_1 = k'_1$ and $\xi^i_2 = \eta^\alpha_2\,B^i_\alpha$.

Hence, it shows that the condition is necessary for an asymptotic line of order 1. Thus the theorem is true for $p = 1$, and hence for $p = 2, 3, \cdots, n-2$.

(ii) Now we have to show that (2.2) is sufficient. Since $\eta^{\alpha}_{r+1} B^{i}_{\alpha}$ are all tangential to F_{n-1} and thus (2.2) implies that ξ^{i}_{r+1} $(r = 1, 2, \cdots, p)$ are all tangential to F_n.

(iii) Lastly, we show that (2.3) is sufficient. So let us assume that (2.3) is sufficient to define an asymptotic line of order p. Then $k_s = k'_s$ $(s = 1, 2, 3, \cdots, p+1)$ can be considered as defining an asymptotic line of order p, with additional property

$$k_{p+1} = k'_{p+1}. \tag{2.9}$$

Hence (2.2) and (2.3) are satisfied and therefore also (2.4). From (2.4), we have

$$g_{ij}\, k^{2}_{p+1}\, \xi^{i}_{p+2}\, \xi^{j}_{p+2} = g_{ij}\,(k'_{p+1}\, \eta^{\alpha}_{p+2}\, B^{i}_{\alpha} + J^{i}_{\alpha\beta}\, \eta^{\alpha}_{p+1}\, \eta^{\beta}_{1}) \times$$
$$\times\, (k'_{p+1}\, \eta^{\delta}_{p+2}\, B^{i}_{\delta} + J^{j}_{\gamma\varepsilon}\, \eta^{\gamma}_{p+1}\, \eta^{\varepsilon}_{1})$$

which on simplification gives

$$k^{2}_{p+1} = k'^{2}_{p+1} + g_{ij}\, J^{i}_{\alpha\beta}\, \eta^{\alpha}_{p+1}\, \eta^{\beta}_{1}\, J^{j}_{\gamma\varepsilon}\, \eta^{\gamma}_{p+1}\, \eta^{\varepsilon}_{1}.$$

The above relation due to (2.9) reduces to

$$J^{i}_{\alpha\beta}\, \eta^{\alpha}_{p+1}\, \eta^{\beta}_{1} = 0. \tag{2.10}$$

Substituting (2.9) and (2.10) in (2.4), we get $\xi^{i}_{p+2} = \eta^{\alpha}_{p+2} B^{i}_{\alpha}$ which together with (2.2), gives the result for asymptotic line of order $p+1$.

Hence, from (ii), the curve is an asymptotic line of order $p+1$. But it is seen from (2.8) that the condition (2.3) for $r = 1$, is sufficient to define an asymptotic line of order 1; so that the theorem is true for $p = 1$, and hence for $p = 2, 3, \cdots, n-2$.

COROLLARY (2.1). The necessary and sufficient condition for an asyptotic line of order p is

$$J^{i}_{\alpha\beta}\, \eta^{\alpha}_{r}\, \eta^{\beta}_{1} = 0 \qquad (r = 1, 2, 3, \cdots, p). \tag{2.11}$$

This can be written in an equivalent form

$$\tilde{\Omega}_{\alpha\beta}\, \eta^{\alpha}_{r}\, \eta^{\beta}_{1} = 0 \qquad (r = 1, 2, 3, \cdots, p) \tag{2.12}$$

due to the fact that it will certainly be an asymptotic line of order 1.

3. EXPRESSION FOR $(n-1)$*th* CURVATURE OF A CURVE IN F_n

We now consider any asymptotic line of highest order (viz. $n-2$). Since in this case the vectors η^{α}_{r} $(r = 1, 2, 3, \cdots, n-1)$ are all tangential to F_{n-1} it follows that the remaining normal η^{α}_{n} is orthogonal to F_{n-1}.

The first $(n-2)$ curvatures in F_n are given by

$$k_r = k'_r \qquad (r = 1, 2, \cdots, n-2).$$

We can also find k_{n-1} at any point of the curve. By differentiating the relation

$$\xi^i_{n-1} = \eta^\alpha_{n-1} B^i_\alpha$$

along the curve, we have

$$\frac{\delta}{\delta s} \xi^i_{n-1} = B^i_\alpha \frac{\delta}{\delta s} \eta^\alpha_{n-1} + \eta^\alpha_{n-1} \frac{\delta}{\delta s} B^i_\alpha$$

which with the help of Frenet's formulae for F_n and F_{n-1} and (1.7) becomes

$$-k_{n-2}\, \xi^i_{n-2} + k_{n-1}\, \xi^i_n = -k'_{n-2}\, \eta^\alpha_{n-2} B^i_\alpha + J^i_{\alpha\beta}\, \eta^\alpha_{n-1}\, \eta^\beta_1 .$$

By virtue of (2.2) and (2.3) it reduces to

(3.1) $$k_{n-1}\, \xi^i_n = J^i_{\alpha\beta}\, \eta^\alpha_{n-1}\, \eta^\beta_1 .$$

Multiplying (3.1) by $\underset{n}{\xi_i}$, we have

(3.2) $$k_{n-1} = J^i_{\alpha\beta}\, \eta^\alpha_{n-1}\, \eta^\beta_1\, \underset{n}{\xi_i} .$$

We can eliminate η^α_{n-1} and $\underset{n}{\xi_i}$ from this expression and get a form which involves only a knowledge of the point and the direction of the curve at the point.

Since, at any point, ξ^i_r $(r = 1, 2, 3, \cdots, n)$ are n mutually orthogonal unit vectors in F_n, so we have

(3.3) $$\sum_{r=1}^{n} \underset{r}{\xi_i}\, \underset{r}{\xi_j} = g_{ij}$$

and since η^α_s $(s = 1, 2, 3, \cdots, n-1)$ are $n-1$ mutually orthogonal unit vectors in F_{n-1}

(3.4) $$\sum_{s=1}^{n-1} \eta^\alpha_s\, \eta^\beta_s = g^{\alpha\beta}.$$

As ξ^i_{n-1} is orthogonal to ξ^i_n, we have from (3.1)

$$\underset{n-1}{\xi_i}\, J^i_{\alpha\beta}\, \eta^\alpha_{n-1}\, \eta^\beta_1 = 0 .$$

This relation on combining with (2.10), which is true for $p = 1, 2, 3, \cdots, n-2$, in our case, gives

(3.5) $$\underset{r}{\xi_i}\, J^i_{\alpha\beta}\, \eta^\alpha_{n-1}\, \eta^\beta_1 = 0 .$$

Thus, from (3.2), we have

$$k_{n-1}^2 = (\underset{n}{\xi_i}\, J^i_{\alpha\beta}\, \eta^{\alpha}_{n-1}\, \eta^{\beta}_1)\,(\underset{n}{\xi_j}\, J^j_{\theta\varepsilon}\, \eta^{\theta}_{n-1}\, \eta^{\varepsilon}_1) =$$

$$= \sum_{r=1}^{n} \underset{r}{\xi_i}\, \underset{r}{\xi_j}\, J^i_{\alpha\beta}\, J^j_{\theta\varepsilon}\, \eta^{\alpha}_{n-1}\, \eta^{\theta}_{n-1}\, \eta^{\beta}_1\, \eta^{\varepsilon}_1$$

due to (3.5). From (3.3) and (3.4), we have

$$k_{n-1}^2 = g_{ij}\, J^i_{\alpha\beta}\, J^j_{\theta\varepsilon} \left(g^{\alpha\theta} - \sum_{s=1}^{n-2} \eta^{\alpha}_s\, \eta^{\theta}_s \right) \eta^{\beta}_1\, \eta^{\varepsilon}_1 .$$

As in our case, $p = n - 2$, the equation (2.11) is valid for $p = 1, 2, 3, \cdots, n-2$ and due to this, the above relation becomes

$$k_{n-1}^2 = g_{ij}\, g^{\alpha\theta}\, J^i_{\alpha\beta}\, J^j_{\theta\varepsilon}\, \eta^{\beta}_1\, \eta^{\varepsilon}_1$$

which on substitution from (1.3), (1.4), (1.5), (1.7) and (2.6) gives

$$k_{n-1}^2 = \tilde{\Omega}_{\alpha\beta}\, \tilde{\Omega}^{\alpha}_{\varepsilon}\, \eta^{\beta}_1\, \eta^{\varepsilon}_1 . \tag{3.6}$$

The Gauss equation referred to the intrinsic curvature tensor $\hat{K}_{\alpha\beta\gamma\varepsilon}$ of the hypersurface F_{n-1} is given by [3]

$$\begin{aligned} \hat{K}_{\alpha\beta\gamma\varepsilon} = {} & K_{ijhk}\, B^{ijhk}_{\alpha\beta\gamma\varepsilon} + \Big[(\tilde{\Omega}_{\alpha\gamma}\, \tilde{\Omega}_{\beta\varepsilon} - \tilde{\Omega}_{\alpha\varepsilon}\, \tilde{\Omega}_{\beta\gamma}) + \\ & + 2\, M_{jl}\, B^j_{\beta}\, (\tilde{\Omega}_{\alpha\gamma}\, J^l_{\sigma\varepsilon} - \tilde{\Omega}_{\alpha\varepsilon}\, J^l_{\sigma\gamma})\, \eta^{\sigma}_1 + (\Lambda_{\sigma\beta\gamma}\, \Lambda^{\sigma}_{\alpha\varepsilon} - \Lambda_{\sigma\beta\varepsilon}\, \Lambda^{\sigma}_{\alpha\gamma}) + \\ & + (\Lambda_{\alpha\beta\gamma|\varepsilon} - \Lambda_{\alpha\beta\varepsilon|\gamma}) + g_{ij}\, B^j_{\beta}\, \frac{\partial \Gamma^{*i}_{hk}}{\partial x'^h}\, B^h_{\alpha}\, (J^l_{\sigma\varepsilon}\, B^k_{\gamma} - J^l_{\sigma\gamma}\, B^k_{\varepsilon})\, \eta^{\sigma}_1 \Big] . \end{aligned} \tag{3.7}$$

The term in square brackets can be regarded as the 'relative curvature tensor of F_{n-1} in F_n'. Let us denote it by $L_{\alpha\beta\gamma\varepsilon}$.

The relative curvature of F_{n-1} for the orientation determined by two orthogonal unit vectors λ^{α}_1, λ^{α}_2 (in F_{n-1}) is $L_{\alpha\beta\gamma\varepsilon}\, \lambda^{\alpha}_1\, \lambda^{\beta}_2\, \lambda^{\gamma}_1\, \lambda^{\varepsilon}_1$. It is the difference of the Riemannian curvature of F_{n-1} for that orientation and the component of the Riemannian curvature of F_n for the orientation determined by the same two vectors regarded as in F_n.

By analogy with the mean curvature of F_{n-1} in a given direction, we shall define the mean relative curvature of F_{n-1} in a direction of a point as $1/(n-2)$ times the sum of the relative curvatures for the $(n-2)$ orientations determined by the given direction and $n-2$ mutually orthogonal direction in F_{n-1} at each point, each orthogonal to the given direction. If λ^{α}_1 is the unit vector in the given direction and λ^{α}_r $(r = 2, 3, \cdots, n-1)$ are unit vectors in the other $(n-2)$ directions, the mean relative curvature in the direction of λ^{α}_1 is

$$\frac{1}{n-2} \sum_{r=2}^{n-1} L_{\alpha\beta\gamma\varepsilon}\, \lambda^{\alpha}_1\, \lambda^{\beta}_r\, \lambda^{\gamma}_1\, \lambda^{\varepsilon}_r .$$

Since $L_{\alpha\beta\gamma\varepsilon}$ is skew symmetric in γ, ε, the above expression becomes equal to

$$-\frac{1}{n-2}\sum_{r=2}^{n-1} L_{\alpha\beta\gamma\varepsilon}\,\lambda_1^\alpha\,\lambda_1^\varepsilon\,\lambda_r^\beta\,\lambda_r^\gamma$$

which due to (3.4) reduces to

$$-\frac{1}{n-2}\,g^{\beta\gamma}\,L_{\alpha\beta\gamma\varepsilon}\,\lambda_1^\alpha\,\lambda_1^\varepsilon\,.$$

Thus, the mean relative curvature for an asymptotic direction η_1^α becomes

$$-\frac{1}{n-2}\,g^{\beta\gamma}\,L_{\alpha\beta\gamma\varepsilon}\,\eta_1^\alpha\,\eta_1^\varepsilon\,. \tag{3.8}$$

With the help of equations (1.6), (1.10), (2.6), (2.7) and (3.6), the expression (3.8) for the mean relative curvature for an asymptotic direction η_1^α reduces to

$$k_{n-1}^2 = -(n-2)\,H\,. \tag{3.9}$$

The symbol H in the above relation stand for the mean relative curvature of F_{n-1} in F_n in the direction of the curve.

4. SOME SPECIAL CASES

(i) a hypersurface F_{n-1} in a Finsler space of constant curvature F_n^*.

If R is the Riemannian curvature of F_n^*, then we have

$$K_{ihk}^l\,\xi_1^i = R\,(\underset{1}{\xi_h}\,\delta_k^l - \underset{1}{\xi_k}\,\delta_h^l)\,. \tag{4.1}$$

Thus the mean intrinsic curvature in the direction of η_1^α, equal to

$$-\frac{1}{n-2}\,g^{\beta\gamma}\,K_{ijhk}\,B_{\alpha\beta\gamma\varepsilon}^{ijhk}\,\eta_1^\alpha\,\eta_1^\varepsilon$$

becomes equal to R, with the help of (1.11) and (4.1). And also the quantity $-\frac{1}{n-2}\,g^{\beta\gamma}\,\widehat{K}_{\alpha\beta\gamma\varepsilon}\,\eta_1^\alpha\,\eta_1^\varepsilon$ becomes

$$\frac{1}{n-2}\,\widehat{K}_{\alpha\varepsilon}\,\eta_1^\alpha\,\eta_1^\varepsilon = J' \tag{4.2}$$

where J′ is the Riemannian curvature of F_{n-1} in the direction of the curve. Thus from equations (3.7), (3.9), (4.1) and (4.2), we have

$$k_{n-1}^2 = (n-2)\,(R - J')\,. \tag{4.3}$$

(ii) a hypersurface of constant Riemannian curvature.

In this case $J' = R'$ for every direction, where R′ is the constant Riemannian curvature of F_n^*, so the equation (4.3) reduces to

$$k_{n-1}^2 = (n-2)\,(R - R')\,.$$

These properties can easily be considered for subspace of Finsler spaces, and there will be no change in the conditions for asymptotic lines of order p by taking the induced covariant derivative, as the induced and intrinsic covariant derivative along the tangent to the curve is the same as remarked by Rund [3].

REFERENCES

[1] H. A. HAYDEN, *Asymptotic lines in a* V_m *in a* V_n. « Proc. London Math. Soc. », (2) *33*, 22–31 (1932).

[2] H. RUND, *The differential geometry of Finsler spaces*. Springer-Verlag Berlin-Heidelberg-Gottingen, 1959.

[3] H. RUND, *The intrinsic and induced curvature theories of subspaces of a Finsler space*. Tensor (N.S.), *16*, 294–312 (1965).