Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Chung-Chun Yang

On the growth of the composition of entire and meromorphic functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **51** (1971), n.1-2, p. 36–40.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_51_1-2_36_0>

Funzioni meromorfe. — *On the growth of the composition of entire and meromorphic functions.* Nota (*) di CHUNG–CHUN YANG (**), presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia $f(z)$ una funzione meromorfa non costante, $g(z)$ una trascendente intera di ordine finito, e $p(z)$ un polinomio. L'accrescimento di una funzione meromorfa f si esprime mediante la caratteristica di Nevanlinna $T(r, f)$. Si dimostra che il rapporto $T(r, f(g))/T(r, f(p)) \to \infty$ quando $r \to \infty$. Si presenta un'applicazione dei risultati.

1. INTRODUCTION

Let $f(z)$ be a meromorphic function in the plane $|z| < \infty$. The growth of f is measured by the Nevanlinna characteristic function $T(r, f)$. The order ρ of f is defined as follows:

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r} .$$

Here and in the sequel it is assumed that the reader is familiar with Nevanlinna's first and second fundamental theorems. In particular the reader is assumed to be familiar with the standard notations such as $T(r, f)$, $N(r, f)$, $N(r, a, f)$, etc. (see e.g. [4]).

The following results have been established about the growth of the composition of entire and meromorphic functions.

THEOREM A (Clunie [4, p. 54]). *If f and g are transcendental and entire, then*

$$\lim_{r\to\infty} T(r, f(g))/T(r, f) = \infty . \tag{1}$$

Recently, Gross obtained an extension of Nevanlinna's second fundamental theorem and used it to prove (1) for a large class of meromorphic functions as follows.

THEOREM B (Gross [3, p. 1031]). *For any sequence $a_1, a_2 \cdots |a_i| \le |a_{i+1}|$, let $\delta(k)$ = minimum of the distances between the first k points of the sequence. Let f be meromorphic, transcendental and such that for three distinct numbers* A_i, $i = 1, 2, 3$

$$\delta(n(r, f^{-1}(A_i))) \ge r^{-t} \tag{2}$$

(*) Pervenuta all'Accademia il 9 gosto 1971.

(**) Mathematics Research Center, Naval Research Laboratory, Washington, D.C. 20390.

for some t and for all r outside a set of finite measure. If g is transcendental and such that $g(0) \notin f^{-1}(A_i)$; $i = 1, 2, 3$, *then* (1) *holds as r approaches infinity outside a set of finite measure.*

It was also stated in [3] that by a similar but somewhat more elaborate argument one could prove the following more general result.

THEOREM C (Gross [3, p. 1032]). *If f and g are as in Theorem B and p is any polynomial, then*

$$T(r, f(g))/T(r, f(p)) \to \infty \tag{3}$$

as $r \to \infty$ *outside a set of r of finite measure.*

Using Theorem C one can easily prove the following result.

THEOREM D (Gross [3, p. 1032]). *Let f and p be as in Theorem C and g be entire and such that for three distinct numbers* A_i, $i = 1, 2, 3$

$$g(0) \notin f^{-1}(A_i); \qquad i = 1, 2, 3. \tag{4}$$

Suppose that the following identity holds:

$$f(g) = f(p). \tag{5}$$

Then g is *a* polynomial of the same degree as p.

In this paper we shall show that for an arbitrary transcendental meromorphic function f and entire function g of finite order the conclusions of Theorems C and D still hold even without the geometric restrictions (2) and (4) on the inverse images of f and g, respectively.

The following two theorems are our main results.

THEOREM 1. *If f is a non-constant meromorphic function in the plane, g is transcendental, entire, and of finite order, and p is a polynomial, then*

$$\varliminf_{r \to \infty} \frac{T(r, f(g))}{T(r, f(p))} = \infty. \tag{6}$$

Remarks. (i) The assumption that g is of finite order is crucial since Clunie [2] showed that there exist a transcendental meromorphic function f and a transcendental entire function g of infinite order such that

$$\varliminf_{r \to \infty} T(r, f(g))/T(r, f) = 0.$$

(ii) Theorem 1 extends a result of Clunie [2, p. 78] which dealt with the case $p(z) \equiv z$.

As an application of Theorem 1 we have, the following:

THEOREM 2. *Let f be a non-constant meromorphic function, g an entire function of finite order, and p a polynomial. Suppose that* (5) *holds. Then g must be a polynomial of the same degree as p.*

2. Preliminary lemmas

In this section we quote the following lemmas which will be used later.

Lemma 1 (Ahlfors, see e.g. [5]). *If $f(z)$ is meromorphic in the plane, then for all complex numbers c outside a set of zero capacity, depending on f,*

$$\mathrm{N}\left(r, \frac{1}{f-c}\right) \sim \mathrm{T}(r, f) \quad \text{as} \quad r \to \infty. \tag{7}$$

This lemma enables us to compare the growth of two functions by examining their associated N *functions.*

Lemma 2 (Valiron [6]). *Let g be an entire function of finite order which is not a polynomial. Denote the maximum modulus of g on $|z| = r$ by $\mathrm{M}(r) = \mathrm{M}(r, g)$. Let $t = t(|\omega|)$ be the inverse function of $|\omega| = \mathrm{M}(t, g)$. Then, given $\varepsilon > 0$, there exists a constant $\mathrm{A}(\varepsilon)$ such that the equation $g(z) = \omega$ has a solution in*

$$|z| < t^{1+\varepsilon} \tag{8}$$

provided $|\omega| > \mathrm{A}(\varepsilon)$.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. First we may choose, according to Lemma 1, a complex number c such that as r approaches infinity

$$\mathrm{N}\left(r, \frac{1}{f(g)-c}\right) \sim \mathrm{T}(r, f(g)) \tag{9}$$

and

$$\mathrm{N}\left(r, \frac{1}{f(p)-c}\right) \sim \mathrm{T}(r, f(p)). \tag{10}$$

Now since $g(z)$ is of finite order, and to each ω with $|\omega| > \mathrm{A}(\varepsilon)$ ($\mathrm{A}(\varepsilon)$ as in Lemma 2) there corresponds at least one t and an argument θ_ω such that

$$\omega = \mathrm{M}(t, g)\, e^{i\theta_\omega}, \tag{11}$$

and by Lemma 2 to each ω_0 in the annulus $\mathrm{A}(\varepsilon) \leq |\omega_0| = |\omega|$ one can always find at least one z_0 in $|z| \leq t^{1+\varepsilon}$ (ε is a fixed positive constant) such that $g(z_0) = \omega_0$.

Assume that the degree of p is $m (\geq 1)$. Choose k an arbitrary fixed large positive number and n a fixed number such that $n > k(m+1)$. Let $\{r_i\}$ be an arbitrary sequence of positive numbers with $r_0 > \mathrm{A}(\varepsilon)$, $r_i < r_{i+1}$ and $r_i \uparrow \infty$, $i = 0, 1, 2, \cdots$. Then if $\mathrm{A}(\varepsilon) < r_0 < |\omega| \leq r_i$,

$M(t_i, g) = |\omega|$, and since g is transcendental, it follows that for sufficiently large $|\omega|$ we have

$$t_i \leq r_i^{1/n}. \tag{12}$$

Thus the image of the disc $|z| \leq r_i$ under the map g will cover the annulus

$$r_0^n < |\omega| \leq r_i^n. \tag{13}$$

Now we examine the preimages of c points of $f(g)$ and $f(p)$ respectively.

It follows from (13) that

$$N\left(r_i, \frac{1}{f(g)-c}\right) \geq N\left(r_i^n, \frac{1}{f-c}\right) - N\left(r_0^n, \frac{1}{f-c}\right) \tag{14}$$

or

$$N\left(r_i, \frac{1}{f(g)-c}\right) \geq N\left(r_i^n, \frac{1}{f-c}\right) - O(1). \tag{15}$$

On the other hand it is easy to verify that

$$N\left(r_i, \frac{1}{f(p)-c}\right) \leq N\left(r_i^{m+1}, \frac{1}{f-c}\right) + O(1). \tag{16}$$

From (9), (10), (15) and (16), we can get

$$\lim_{\overline{i\to\infty}} \frac{T(r_i, f(g))}{T(r_i, f(p))} \sim \lim_{\overline{i\to\infty}} \frac{N\left(r_i, \dfrac{1}{f(g)-c}\right)}{N\left(r_i, \dfrac{1}{f(p)-c}\right)} \geq \tag{17}$$

$$\geq \lim_{\overline{i\to\infty}} \frac{N\left(r_i^n, \dfrac{1}{f-c}\right) - O(1)}{N\left(r_i^{m+1}, \dfrac{1}{f-c}\right) + O(1)} \sim$$

$$\sim \lim_{\overline{i\to\infty}} \frac{N\left(r_i^n, \dfrac{1}{f-c}\right)}{N\left(r_i^{m+1}, \dfrac{1}{f-c}\right)}.$$

(since $N(r, 1/(f-c)) \to \infty$, we can drop the bounded term $O(1)$ in the inequality without affecting the limit).

However,

$$\frac{N\left(r_i^n, \dfrac{1}{f-c}\right)}{N\left(r_i^{m+1}, \dfrac{1}{f-c}\right)} = \left(\frac{N\left(r_i^n, \dfrac{1}{f-c}\right)}{\log r_i^n}\right) \Bigg/ \left(\frac{N\left(r_i^{m+1}, \dfrac{1}{f-c}\right)}{\log r_i^{m+1}}\right) \cdot \frac{\log r_i^n}{\log r_i^{m+1}}. \tag{18}$$

Now $N'(r, f)$, as follows immediately from its definition, is a convex increasing function of $\log r$ so the first factor of the right hand side of equation (18) is greater than or equal to 1.

Hence we have from (18) that

$$\lim_{\overline{i\to\infty}} \frac{N\left(r_i^n, \frac{1}{f-c}\right)}{N\left(r_i^{m+1}, \frac{1}{f-c}\right)} \geq \frac{n \log r_i}{(m+1)\log r_i} \geq \frac{n}{m+1} > k. \tag{19}$$

As k can be arbitrarily large, our assertion follows from this and (17).

Proof of Theorem 2. According to Theorem 1, we see right away that if (5) holds then g must be a polynomial. That g has the same degree as p follows from a result of Baker and Gross [1]. The theorem is thus proved.

4. Open question

We do not know if the assumption that g is of finite order is necessary for Theorem 2. Thus, at the end of this paper, we would like to ask the following question.

Question. Let f be a non-constant meromorphic function, g an entire function, and p a polynomial. Suppose that $f(g) = f(p)$. Must g also be a polynomial of the same degree as p?

References

[1] I. N. Baker and F. Gross, *On factorizing entire functions*, « Proc. Lond. Math. Soc. », (3) *18*, 69–76 (1968).

[2] J. Clunie, *The composition of entire and meromorphic functions*, Mathematical essays dedicated to A. J. MacIntyre, Ohio University Press, 1970.

[3] F. Gross, *On the growth of f(g)*, « Bull. Amer. Math. Soc. », *74*, (5), 1030–1033 (1968).

[4] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monograph, Clarendon Press, Oxford, 1964.

[5] R. Nevanlinna, *Analytic Functions*, Springer-Verlag, New York, 1970, p. 276.

[6] G. Valiron, *Sur un théorème de M. Fatou*, « Bull. Sci. Math. », (2) *46*, 200–208 (1922).