Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Fixed points of multi-valued functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **51** (1971), n.1-2, p. 32–35.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_51_1-2_32_0>

Analisi funzionale. — *Fixed points of multi-valued functions.* Nota (*) di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sono stabiliti vari nuovi Teoremi sul punto fisso per funzioni multivoche definite in un sottoinsieme chiuso convesso di uno spazio localmente convesso, oppure di uno spazio di Banach.

1. INTRODUCTION

Let X be a non-empty closed convex subset of a real Hausdorff topological vector space E. Let F assign to each $x \in X$ a non-empty closed convex subset $F(x)$ of E. If $x \in X$ we shall denote the set $\{z \in E : z = x + c(y - x)$ for some $y \in X$ and $c \geq 0\}$ by $I_X(x)$. More generally, if g is a multi-valued function defined on X such that for each $x \in X$ $g(x)$ is a non-empty subset of E, we put $I_X(g(x)) = \{z \in E : z = u + c(y - x)$ for some $u \in g(x)$, $y \in X$ and $c \geq 0\}$. If $S \subset E$, cl(S), cocl(S), int(S) and bd(S) will stand for the closure, convex closure, interior and boundary of S, respectively. In the sequel we shall consider the following conditions which may be imposed on F:

(1) $F(x) \cap I_X(x) \neq \varnothing$ for each $x \in X$,

(2) $F(x) \cap \mathrm{cl}(I_X(x)) \neq \varnothing$ for each $x \in X$,

(3) $\mathrm{int}(X) \neq \varnothing$ and for some $w \in \mathrm{int}(X)$, $z - w \neq m(y - w)$ for all $y \in \mathrm{bd}(X)$, $z \in F(y)$ and $m > 1$,

(4) If the net $\{x_d : d \in D\} \subset X$ converges to x, and the net $\{y_d : d \in D\}$, where $y_d \in F(x_d)$ for each $d \in D$, converges to y, then $F(x) \cap \{z \in E : z = x + c(y - x)$ for some $c \geq 0\} \neq \varnothing$,

(5) If $x \in X$ and h is a continuous linear functional which strictly separates x and $F(x)$, then there exists a neighborhood N of x in X such that h strictly separates N and $\cup\{F(y) : y \in N\}$.

If x is an internal point of X then (1) certainly holds. If the interior of X is non-empty and F happens to be single-valued, then (2) implies (3), but not vice versa. If F is upper semi-continuous, or even merely upper demi-continuous ([2], p. 236), then it satisfies (5).

Fan ([2], p. 238) proved in fact the following result:

THEOREM A. *Let* X *be a non-empty compact convex subset of a real, locally convex, Hausdorff topological vector space* E. *Let* F *assign to each* $x \in X$ *a non-empty closed convex subset of* E. *If* F *satisfies both* (1) *and* (5), *then it has a fixed point.*

Glebov ([3], p. 447) established the following Theorem, after introducing condition (4).

(*) Pervenuta all'Accademia il 26 agosto 1971.

THEOREM B. *Let* X *be a non-empty compact convex subset of a real, locally convex, Hausdorff topological vector space* E. *Let* F *assign to each* $x \in X$ *a non-empty closed convex subset of* X. *If* F *satisfies* (4), *then it has a fixed point.*

Cellina ([1], p. 30) observed that Theorem B is a consequence of Theorem A because (4) implies (5) when the range of F is contained in a compact subset of E. He also showed that if $F(x)$ is compact for each $x \in X$, then (5) implies (4).

In this Note we intend to elaborate Cellina's observation and to extend both Theorems.

2. COMPACT DOMAINS

THEOREM 1. *Let* X *be a non-empty compact convex subset of a real Hausdorff topological vector space* E. *Let* f *assign to each* $x \in X$ *a non-empty subset of* E *and let* g *assign to each* $x \in X$ *a non-empty compact subset of* E. *Suppose that* f *and* g *enjoy the following property*:

(6) If a continuous linear functional h strictly separates $f(x)$ and $g(x)$ where $x \in X$, then there exists a neighborhood N of x in X such that h strictly separates $\cup\{f(y) : y \in N\}$ and $\cup\{g(y) : y \in N\}$.

If

(7) $f(x) \cap \mathrm{cl}(I_X(g(x))) \neq \varnothing$ for each $x \in X$,

then there is a point $y \in X$ for which $f(y)$ and $g(y)$ cannot be strictly separated by a continuous linear functional.

Proof. Assuming our claim is false, we follow in Fan's footsteps in his proof of Theorem 5 of [2] and find a point $y \in X$ and a continuous linear functional h such that $h(u-v) < 0$ for all $u \in f(y)$, $v \in g(y)$ and $h(x-y) \geq 0$ for all $x \in X$. A certain net $\{z_d : d \in D\} \subset I_X(g(y))$ converges to some $z \in f(y)$. By definition, $z_d = w_d + c_d(x_d - y)$ where $\{w_d : d \in D\} \subset g(y)$, $\{x_d : d \in D\} \subset X$ and $c_d \geq 0$ for each $d \in D$. We may assume that $\{w_d : d \in D\}$ converges to some $w \in g(y)$. But $h(z_d - w_d) = c_d h(x_d - y) \geq 0$ for each $d \in D$. Hence $h(z - w) \geq 0$, a contradiction.

LEMMA 1. *If* f *and* g *satisfy*

(8) *If the nets* $\{x_d : d \in D\}$, $\{u_d : d \in D\}$ *and* $\{v_d : d \in D\}$ *converge to* x, u, v *respectively, where* $x_d \in X$, $u_d \in f(x_d)$ *and* $v_d \in g(x_d)$, *then* $f(x) \cap \{z \in E : z = v + c(u - v)$ *for some* $c \geq 0\} \neq \varnothing$,

and the ranges of f *and* g *are contained in compact subsets of* E, *then they satisfy* (6).

Proof. Assume that our assertion is not true for a certain $x \in X$. Then, although there exists a continuous linear functional h such that $h(p - q) > 0$ for all $p \in f(x)$ and $q \in g(x)$, for each neighborhood N of x in X and for each $e > 0$ we can find $u(N, e) \in f(N)$ and $v(N, e) \in g(N)$ such that $h(u(N, e)) - h(v(N, e)) \leq e$. We can assume that the nets $\{u(N, e)\}$ and $\{v(N, e)\}$ converge to u and v respectively. There is a point $p \in f(x)$ such that $p = v + c(u - v)$ for some $c \geq 0$. This implies that $h(p) = h(v) + c(h(u) - h(v)) \leq h(v)$, a contradiction.

COROLLARY 1. *Let* X *be a non-empty compact convex subset of a real Hausdorff locally convex topological vector space* E. *Let* F *assign to each* $x \in X$ *a non-empty closed convex subset of* E. *If* E *satisfies* (5) *and* (2), *then it has a fixed point.*

Proof. A point and a closed convex set which does not contain it can be strictly separated by a continuous linear functional.

This result generalizes a Theorem of Halpern's ([4], p. 88).

COROLLARY 2. *Let* X *be a non-empty compact convex subset of a real Hausdorff locally convex topological vector space* E. *Let* F *assign to each* $x \in X$ *a non-empty closed convex subset of* E. *If the range of* F *is contained in a compact subset of* E *and if it satisfies* (2) *and* (4), *then it has a fixed point.*

Proof. This assertion is implied by Lemma 1, as well as by Cellina's original observation.

3. NON-COMPACT DOMAINS

Let (Y, d) be a metric space and let $P(Y) = \{A : A \text{ is a non-empty subset of } Y\}$. If $B \subset Y$ is bounded we put (cfr. [5], p. 412) $a(B) = \inf \{r > 0 : B$ can be covered by a finite number of sets of diameter less than or equal to $r\}$. Now let S be a non-empty subset of Y and let F assign to each $s \in S$ a non-empty subset of Y. We associate with F another function $G_F : P(S) \to P(Y)$ by $G_F(A) = \cup \{F(a) : a \in A\}$ where $A \in P(S)$. F will be called condensing if for every bounded subset B of S with $a(B) > 0$ $G_F(B)$ is bounded and $a(G_F(B)) < a(B)$. If in addition $a(B) = 0 \Rightarrow a(G_F(B)) = 0$, F will be called strongly condensing.

Another definition of a "measure of non-compactness" is feasible, namely $b_Y(B) = \inf \{r > 0 : B$ can be covered by a finite number of balls with centers in Y and radius $r\}$. This leads to a treatment of condensing functions in locally convex spaces. This treatment enables us to state the following results in a locally convex setting. However, we prefer, for the sake of simplicity, to restrict our attention to Banach spaces.

The idea of the following lemma is due to Halpern ([4], p. 90).

LEMMA 2. *Let* X *be a convex subset of a vector space* E. *Let* z *belong to* X *and let* y *belong to* $I_X(z)$. *Let* C *be a convex subset of* E *which contains* y. *If* $z \in K = X \cap C$, *then* $y \in I_K(z)$.

Proof. It is not difficult to see that $y = z + c(u - z)$ for some $c \geq 1$ and $u \in X$. The convexity of C implies that $u \in C$, so that $y \in I_K(z)$.

Reinermann ([7], p. 341) used the next lemma, a direct consequence of Zorn's lemma.

LEMMA 3. *Let* g *be a self-mapping of a partially ordered set* T *every chain of which has an upper bound. If* $x \leq g(x)$ *for all* $x \in T$, *then* g *has a fixed point.*

Let X be a non-empty convex subset of a vector space E. Let $F : X \to P(E)$ satisfy (1). For each $x \in X$ we choose once and for all a point $u \in X$ and a point $z \in F(x)$ such that $z = x + c(u - x)$ with $c \geq 1$. If $Y \subset X$ and $x \in Y$, F will be said to satisfy (1) with the original coordinates if this u belongs to Y.

THEOREM 2. *Let* X *be a non-empty closed convex subset of a Banach space* E. *Let a strongly condensing multi-valued function* F *assign to each* $x \in X$ *a non-empty closed convex subset of* E. *If* F *satisfies* (1), (4) *and has a bounded range, then it has a fixed point.*

Proof. Fix a point $w \in X$ and consider the following non-empty family of non-empty subsets of $X : T = \{Y \subset X : w \in Y$, F satisfies (1) on Y with the original coordinates, Y is closed and convex$\}$. Define a function $g : T \to T$ by $g(Y) = \text{cocl}(G_F(Y) \cup w) \cap Y$ where $Y \in T$. With the aid of Lemma 2 it is seen that g and T satisfy the conditions of Lemma 3 when T is ordered by inclusion. Since F is condensing Z, the fixed point of g, is totally bounded. Since this fixed point is also closed and E is complete, it is actually compact. $G_F(Z)$ is contained in a compact subset of E because F is strongly condensing. An appeal to Corollary 2 completes the proof.

The following result can be established in a similar fashion.

THEOREM 3. *Let* X *be a non-empty closed convex subset of a Banach space* E. *Let a condensing multi-valued function* F *assign to each* $x \in X$ *a non-empty closed convex subset of* E. *If* F *satisfies* (1), (5) *and has a bounded range, then it has a fixed point.*

Condition (3) appears in our last result. This proposition can be proved by first considering the single-valued case, where the Minkowski functional comes in handy, and then applying a special case of Michael's selection Theorem ([6], p. 1404).

THEOREM 4. *Let* X *be a closed convex subset of a Banach space* E. *Let a lower semi-continuous condensing multivalued function* F *assign to each* $x \in X$ *a non-empty closed convex subset of* E. *If* F *satisfies* (3) *and has a bounded range, then it has a fixed point.*

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