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On an extension of a Theorem due to J. B. Diaz and F. T. Metcalf

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Analisi funzionale. — *On an extension of a Theorem due to J. B. Diaz and F. T. Metcalf* (*). Nota (**) di UGO BARBUTI e SERGIO GUERRA, presentata dal Socio G. SANSONE.

RIASSUNTO. — Un noto risultato di J. B. Diaz e F. T. Metcalf, riguardante la convergenza a punti fissi di traiettorie generate da contrazioni definite su un insieme convesso e chiuso di uno spazio di Banach strettamente convesso, è qui esteso al caso delle contrazioni generalizzate.

Let be (X, δ) a metric space and T a generalized contraction of X into itself, namely a map such that the following disequality holds:

$$\delta(Tx_1, Tx_2) \leq \alpha\delta(x_1, x_2) + \beta[\delta(Tx_1, x_1) + \delta(Tx_2, x_2)], \quad \forall x_1, x_2 \in X, \tag{1}$$

where α, β are two not negative constants such that

$$\alpha + 2\beta \leq 1 \text{ (1)}. \tag{2}$$

In order of such a transformation the problem arises to determine conditions for the convergence of the Picard sequences generated by itself or by other transformations associated to it.

In [1] J. B. Diaz and F. T. Metcalf have proved the following Theorem, which extends a previous result, due to M. Edelstein (2):

If Y *is a convex and closed subset of a strictly convex Banach space* X *and* $T: Y \to Y$ *is a continuous transformation which satisfies the condition* (1) *with* $\beta = 0$ *and* $\alpha = 1$, *namely the condition*

$$\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\|, \qquad \forall x_1, x_2 \in Y \tag{3}$$

and if T(Y) *is contained in a compact* $Y_1 (\subset Y)$ *then, for every* $x \in Y$, *the Picard sequence starting from* x *and generated by the transformation* U_λ:

$$U_\lambda x = \lambda Tx + (1 - \lambda)x, \tag{4}$$

where λ *is such that* $0 < \lambda < 1$, *converges to a fixed point for* T.

The aim of this brief note is to state an extension of this Theorem for the generalized contractions.

(*) Lavoro eseguito nell'ambito delle attività del G.N.A.F.A. del C.N.R.

(**) Pervenuta all'Accademia il 2 luglio 1971.

(1) Such transformations, with $\alpha + 2\beta < 1$ were already studied by various Authors (see [3], [4], [5], [6]).

(2) See [2], Theorem 4, p. 476.

THEOREM. *Remaining valid all the hypotheses of the previous Theorem, the condition* (3) *being replaced by*

$$(3') \qquad \|Tx_1 - Tx_2\| \leq \alpha \|x_1 - x_2\| + \beta (\|Tx_1 - x_1\| + \|Tx_2 - x_2\|), \quad \forall x_1, x_2 \in Y$$

where $\alpha + 2\beta \leq 1$, *then the same thesis holds.*

We can see at once that U_λ is actually defined on Y and $U_\lambda(Y) \subset Y$, as Y is convex; moreover the sets F(T) and $F(U_\lambda)$ coincide for every λ. Moreover F(T) (and therefore $F(U_\lambda)$) is not void for the Schauder's Theorem (3), as Y is convex and closed and T(Y) is contained in a compact. Let us also observe that, for any fixed $x \in Y$, the convex hull generated by $T(Y) \cup \{x\}$ is compact, as the well-known Mazur's Theorem shows (4). Therefore the Picard sequence $\{U_\lambda^n x\}_{n \in N}$ (whose points belong to the convex hull) is compact, that is, U_λ is sequentially compact with respect to every $x \in Y$.

Let us also point out that if $u \in F(T)$, for every $x \in Y$, owing to (3′) and the triangular property of the distance, we have

$$\|Tx - u\| = \|Tx - Tu\| \leq \alpha \|x - u\| + \beta \|Tx - x\| \leq$$
$$\leq \alpha \|x - u\| + \beta \|Tx - u\| + \beta \|x - u\|$$

and as $\beta \leq \frac{1-\alpha}{2} \leq \frac{1}{2}$, it follows that

$$\|Tx - u\| \leq \frac{\alpha + \beta}{1 - \beta} \|x - u\|$$

and therefore, as $\alpha + 2\beta \leq 1$

$$(5) \qquad \|Tx - u\| \leq \|x - u\|.$$

For $x \neq u$ it results

$$(6) \qquad \|U_\lambda x - u\| = \|U_\lambda x - U_\lambda u\| = \|\lambda (Tx - u) + (1 - \lambda)(x - u)\| =$$
$$= \|x - u\| \cdot \left\| \lambda \frac{Tx - u}{\|x - u\|} + (1 - \lambda) \frac{x - u}{\|x - u\|} \right\|.$$

If in the (5) the strict inequality holds, then for the second factor of the last member of the (6), we have

$$(7) \qquad \left\| \lambda \frac{Tx - u}{\|x - u\|} + (1 - \lambda) \frac{x - u}{\|x - u\|} \right\| \leq \lambda \frac{\|Tx - u\|}{\|x - u\|} + (1 - \lambda) < 1.$$

and therefore

$$(8) \qquad \|U_\lambda x - u\| < \|x - u\|.$$

(3) See [7].
(4) See [8].

If, on the contrary, in the (5), the sign of equality holds, and if, as we can always suppose (5), x is not a fixed point, then the two vectors which appear in the first member of (7) have a unitary norm and, as X is strictly convex, (7) is still valid and therefore (8) holds. All the hypotheses of the Theorem 3 of [1] (6) are verified and therefore the thesis holds.

(5) If x is a fixed point, the sequence $[U_\lambda^n x]$ is obviously convergent.
(6) Page 474. See also [9], Theorem 3, p. 75.

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