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A fixed point theorem

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Analisi funzionale. — *A fixed point theorem.* Nota di SIMEON REICH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si migliora un risultato ottenuto da Vasile ed Ana Istrăţescu in una recente Nota lincea [4], relativo all'esistenza di un punto fisso per ogni funzione condensatrice definita su di una palla chiusa in uno spazio di Banach, e se ne fanno alcune applicazioni (1).

1. INTRODUCTION

Let BN(M) be the family of all the bounded nonempty subsets of a metric space (M, d). We define a nonnegative function a on BN(M) by $a(B) = \inf\{r > 0 \mid B$ can be covered by a finite number of sets of diameter less than or equal to $r\}$ where $B \in BN(M)$ (Cf. [6], p. 412).

A function $f: M \to M$ will be called a k-set-contraction where $k \geq 0$ if $a(f(B)) \leq k a(B)$ for all $B \in BN(M)$. It will be called condensing if $a(f(B)) < a(B)$ for all $B \in BN(M)$ such that $a(B)$ is positive.

In the sequel X will denote a Banach space, $B(x_0, r)$, $x_0 \in X$ and $r > 0$, will denote the subset $\{x \in X \mid \|x - x_0\| \leq r\}$, while $S(x_0, r)$ will stand for the subset $\{x \in X \mid \|x - x_0\| = r\}$.

It is known ([1], p. 197) that a continuous condensing function $f: B(x_0, r) \to B(x_0, r)$ has a fixed point.

Recently, Istrăţescu and Istrăţescu ([4], p. 45), although they employed a variant of the function a defined above, have shown in fact that if a continuous k-set-contraction $f: B(0, r) \to X$, $0 \leq k < 1/2$, satisfies $\|x - f(x)\|^2 \geq \|f(x)\|^2 - \|x\|^2$ for $x \in S(0, r)$, then it has a fixed point.

In the next section we present an improvement on this result.

2. A FIXED POINT THEOREM

THEOREM. *Let $f: B(x_0, r) \to X$ be a continuous condensing function. If for every $y \in S(x_0, r)$ there is no $m > 1$ with $f(y) - x_0 = m(y - x_0)$, then f has a fixed point.*

Proof. Define a nonnegative function p on X by

$$p(x) = \begin{cases} 1 & x \in \{x \mid \|f(x) - x_0\| \leq r\} \\ \dfrac{r}{\|f(x) - x_0\|} & x \in \{x \mid \|f(x) - x_0\| \geq r\}. \end{cases}$$

(*) Nella seduta del 18 giugno 1971.

(1) This Note is a proper subset of the author's M. Sc. thesis which is being written now under the supervision of Professors M. Reichaw and P. Saphar at the Department of Mathematics of the Technion–Israel Institute of Technology, Haifa.

The function $g: B(x_0, r) \to B(x_0, r)$ defined by $g(x) = p(x)f(x) + (1 - p(x))x_0$ is continuous and condensing (Cf. [7], p. 11). Therefore it has a fixed point z. Suppose that $\|f(z) - x_0\| > r$. Then $r \dfrac{f(z) - x_0}{\|f(z) - x_0\|} = z - x_0$. Thus $z \in S(x_0, r)$. Since $f(z) - x_0 = r^{-1} \|f(z) - x_0\| (z - x_0)$, it follows that $\|f(z) - x_0\|/r \leq 1$, and the proof is complete.

3. SOME APPLICATIONS

Our first corollary generalizes a result which appears on p. 189 of [3].

COROLLARY 1. *Let* $f: B(x_0, r) \to X$ *be continuous, and suppose that for some* $b \neq 0$ $x + bf(x)$ *is condensing. If for every* $x \in S(x_0, r)$ *there is no* $m > 0$ *such that* $b(f(x) - \bar{y}) = m(x - x_0)$, *then there exists a point* $\bar{x} \in B(x_0, r)$ *such that* $f(\bar{x}) = \bar{y}$.

Proof. The previous Theorem can be applied to

$$h(x) = x + b(f(x) - \bar{y}).$$

COROLLARY 2. *Let* F *be a continuous and condensing mapping of* X *into itself, and put* $f(x) = x - F(x)$. *If for the point* $y_0 \in f(X)$ *there exists a point* $x_0 \in f^{-1}(y_0)$ *such that* $\inf\{N(x_0, r, F)/r \mid 0 < r < \infty\} < 1$ *where* $N(x_0, r, F) = \sup\{\|F(x) - F(x_0)\| \mid x \in S(x_0, r)\}$, *then* $f(X)$ *contains a ball with center* y_0 *and radius* $r_0 \neq 0$.

Proof. For suitable r, r_0 Corollary 1 with $b = -1$ and $\bar{y} \in \dot{B}(y_0, r_0)$ can be applied to $f: B(x_0, r) \to X$.

This is a generalization of a theorem of Reichbach (= Reichaw) ([8], p. 735).

COROLLARY 3. *Let* F *be a continuous and condensing mapping of* X *into itself, and put* $f(x) = x - F(x)$. *If*

$$\inf\Big\{\sup\{\|F(x)\|/\|x\| \mid \|x\| \geq r\}\Big| 0 < r < \infty\Big\} < 1,$$

then $f(X) = X$.

Proof. Let $\bar{y} \in X$ be given. For sufficiently large r Corollary 1 can be applied to f with $b = -1$ and $x_0 = 0$.

This generalizes a result due to Granas ([2], p. 868).

In the next proposition, which extends a result of Kasriel and Nashed ([5], p. 1046), f will denote a function from $B(x_0, r)$ into X such that $F(x) = x - f(x)$ is continuous and condensing, while u will denote an element of X.

COROLLARY 4. *Let* $F_0: B(x_0, r) \to X$ *be continuous and condensing, and let* $f_0(x) = x - F_0(x)$. *Suppose that* $f_0(x_0) = 0$ *and that* $N(x_0, r, F_0)/r < 1$. *Then there exist positive numbers* b, d *with the property that if* $\|(f_0 - f)x\| \leq b$ *for all* $x \in \dot{B}(x_0, r)$ *and* $\|u\| \leq d$, *then the equation* $f(x) = u$ *has a solution in* $B(x_0, r)$.

Proof. Again Corollary 1 with $b = -1$ can be applied to f.

COROLLARY 5 (Cf. [9]). *Let* $f: X \to X$ *be continuous and condensing, and let* $q \in [0, 1]$. *Then either there is an* $x \in X$ *such that* $x = qf(x)$ *or the set* $\{x \mid x = mf(x) \text{ for some } m \in (0, 1)\}$ *is unbounded.*

Proof. Suppose $x = qf(x)$ has no solution in X. Let n be any natural number. Consider the mapping $g: B(0, n) \to X$ defined by $g(x) = qf(x)$. g cannot possess a fixed point. Hence there is $m_n > 1$ and an $x_n \in S(0, n)$ can be found such that $qf(x_n) = m_n x_n$. The result follows.

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