Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A note on automorphisms of Lie algebras

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **51** (1971), n.1-2, p. 1–4.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_51_1-2_1_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1971.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Ferie 1971 (Luglio–Agosto)

(Ogni Nota porta a pie' di pagina la data di arrivo o di presentazione)

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *A note on automorphisms of Lie algebras.* Nota (*) di MARTIN MOSKOWITZ, presentata dal Corrisp. G. ZAPPA.

RIASSUNTO. — Questa Nota presenta un criterio tale che un'algebra di Lie reale o complessa di dimensione finita sia nilpotente nei termini dei valori caratteristici di un automorfismo e fornisce una maggiorazione degli indici n di nilpotenza. Infatti questa maggiorazione è la migliore possibile. Nel caso $n = 2$ si dimostra anche l'inverso e se ne deduce che un gruppo di Lie connesso e semplicemente connesso con $n = 2$ ha sempre una famiglia moltiplicativa $\{\beta_t\}_{t>0}$ degli automorfismi tale che $\lim_{t \to 0} \beta(g) = 1$ per ogni $g \in G$.

In [5] N. Jacobson proves (Theorem 2) that if each eigenvalue λ of a given automorphism α of the finite dimensional Lie algebra $\mathfrak{A}$ is not a root of unity, then $\mathfrak{A}$ is nilpotent. The purpose of this Note is to give a more elementary proof of a slightly less general fact but which, in addition, gives an estimate of the index of nilpotence of $\mathfrak{A}$ in terms of the given data. The converse question was raised in [5] and answered in the negative by J. Dixmier and W. G. Lister in [1] and more recently by Joan Dyer in [2].

THEOREM. *Let $\mathfrak{A}$ be a finite dimensional real or complex Lie algebra and $\alpha \in \mathfrak{a}(\mathfrak{A})$ the group of automorphisms of $\mathfrak{A}$. If each eigenvalue λ of α has modulus > 1 then $\mathfrak{A}$ is nilpotent. Moreover, the index of nilpotence is $\leq \log|\lambda_+|/\log|\lambda_-|$ where $|\lambda_+|$ and $|\lambda_-|$ denote respectively the max and min of moduli of the λ's.*

It should be remarked that this estimate is best possible, i.e., is actually realized in the following example. Let $\mathfrak{A}$ be the Lie algebra of strictly triangular

(*) Pervenuta all'Accademia il 10 agosto 1971.

matrices of order n over the field of real or complex nos. Then, as is well-known, $\mathfrak{N}$ is nilpotent of index $n-1$. Let $\mathfrak{N} = \mathfrak{N}^{(1)} \supset \cdots \supset \mathfrak{N}^{(n-1)} = (0)$ be the descending central series. Choose a basis for $\mathfrak{N}^{(n-2)}$, extend this to a basis for $\mathfrak{N}^{(n-3)}$ etc. and finally to a basis of $\mathfrak{N}$ itself. Define $\alpha_t : \mathfrak{N} \to \mathfrak{N}$ by $(\alpha_t)_{\mathfrak{N}^{(i)}-\mathfrak{N}^{(i-1)}} = t^i I$ where $i = 1, \cdots, n-1$ and t is in the field and $\alpha_t(0) = 0$. Then α_t is a linear map, is diagonalizable with respect to this basis and has for distinct eigenvalues $\{t, t^2, \cdots, t^{n-1}\}$ so that for $t \neq 0$ is invertible. A direct calculation shows α_t is an automorphism. Hence if we take $|t| > 1$ we find that all eigenvalues of α_t have modulus > 1 and $\log|t^{n-1}|/\log|t| = n-1$, the index of nilpotence.

Proof of the Theorem: If $\mathfrak{N}$ were real then by considering its complexification $\mathfrak{N}^c = \mathfrak{N} \otimes_R \mathbf{C}$ it is evidently sufficient to deal only with the complex case. We can therefore apply the following result [Lemma 22, pg. 194, of [3]) which is a generalization of the 3rd Jordan canonical form$/\mathbf{C}$ to Lie algebras:

If $\mathfrak{N}$ is a finite dimensional complex Lie algebra and $\alpha \in \mathfrak{A}(\mathfrak{N})$ then $\alpha = \psi \exp D$ where:

a) $\psi \in \mathfrak{A}(\mathfrak{N})$ and is diagonalizable;

b) D is a nilpotent derivation of $\mathfrak{N}$;

c) D and ψ commute.

Thus $\mathfrak{N} = \sum_{i=1}^{k} V_{\lambda_i}$ where $\lambda_i, \cdots, \lambda_k$ are the distinct eigenvalues of ψ and the V_{λ_i} are the corresponding (geometric) eigenspaces. Since ψ and D commute so do ψ and Exp D. If $x_i \in V_{\lambda_i}$ then $\psi \operatorname{Exp} D(x_i) = \operatorname{Exp} D(\psi(x_i)) = \operatorname{Exp} D(\lambda_i x_i) = \lambda_i \operatorname{Exp} D(x_i)$. Hence $\operatorname{Exp} D(x_i) \in V_{\lambda_i}$. Therefore $\alpha(V_{\lambda_i}) = \psi(\operatorname{Exp} D(V_{\lambda_i})) \subseteq \psi(V_{\lambda_i}) \subseteq V_{\lambda_i}$ so that each V_{λ_i} is an invariant subspace under α (and Exp D). Because D is nilpotent, Exp D is unipotent on all of V and in particular on the invariant subspace V_{λ_i}. Now on V_{λ_i}, $\psi = \lambda_i I$ (in any basis) and $\operatorname{Exp} D = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$ in some basis so that in this basis the restrictions, $(\alpha)_{V_{\lambda_i}} = \begin{pmatrix} \lambda_i & & 0 \\ & \ddots & \\ * & & \lambda_i \end{pmatrix}$ Hence λ_i is an eigenvalue of $(\alpha)_{V_{\lambda_i}}$ and therefore $|\lambda_i| > 1$. Thus, in addition to knowing $|\lambda_i| > 1$ for all i we may assume that α is diagonalizable on $\mathfrak{N}$. It has just been shown that the set of eigenvalues of ψ is contained in those of α. Since the proof of this, essentially, only depended on the facts that $\alpha = \psi \operatorname{Exp} D$, that ψ and D commute and that D is nilpotent and since $\psi = \alpha(\operatorname{Exp} D)^{-1} = \alpha \operatorname{Exp}(-D)$, and $\alpha D = \psi(\operatorname{Exp} D) D = \psi D \operatorname{Exp} D = D\psi \operatorname{Exp} D = D\alpha$ (i.e., α and $-D$ commute) and $-D$ is nilpotent, it follows that the eigenvalues of α and ψ coincide.

Let $x_i \in V_{\lambda_i}$ and $x_j \in V_{\lambda_j}$ where $i, j = 1, \cdots, k$. Then $\alpha[x_i, x_j] = [\alpha(x_i), \alpha(x_j)] = \lambda_i \lambda_j [x_i, x_j]$. It follows that $[V_{\lambda_i}, V_{\lambda_j}] = (0)$ or $\lambda_i \lambda_j$ is an eigenvalue of α. Denote by S_1 the set $\{\lambda_1, \cdots, \lambda_k\}$ of distinct eigenvalues of α, $S_1^n = \{\lambda_{i1}, \cdots, \lambda_{in} : \lambda_{i_s} \in S_1\}$ and $S_n = S_{n-1} \cap S_1^n$ where $n \geq 2$. Then we have

1) $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_2 \supseteq \cdots$
2) $\mathfrak{N} = \mathfrak{N}^{(1)} = \sum_{\lambda_i \in S_1} V_{\lambda_i}$
3) $[V_{\lambda_i}, V_{\lambda_j}] = 0$ if $\lambda_i \lambda_j \notin S_1$
4) $[V_{\lambda_i}, V_{\lambda_j}] \subseteq V_{\lambda_h}$ if $\lambda_i \lambda_j = \lambda_h \in S_1$ i.e., if $\lambda_h \in S_2$.

Therefore by 2), 3) and 4), $\mathfrak{N}^{(2)} = [\mathfrak{N}, \mathfrak{N}] \subseteq \sum_{i,j} [V_{\lambda_i}, V_{\lambda_j}] \subseteq \sum_{\lambda_h \in S_2} V_{\lambda_h}$ and more generally for $n \geq 1$, $\mathfrak{N}^{(n)} \subseteq \sum_{\lambda_h \in S_n} V_{\lambda_h}$ where $\mathfrak{N}^{(n)} = [\mathfrak{N}, \mathfrak{N}^{(n-1)}]$ is the descending central series. To show $\mathfrak{N}$ is nilpotent it suffices to show that $\sum_{\lambda_h \in S_n} V_{\lambda_h} = 0$ for some n. This implies that it is sufficient to show $S_n = \varnothing$ for some n. Suppose all $S_n \neq \varnothing$. Since S_1 is finite and therefore compact it follows from 1) and the finite intersection property that $\bigcap_{n=1}^{\infty} S_n \neq \varnothing$. But $\bigcap_{n=1}^{\infty} S_n = \bigcap_{n=2}^{\infty} S_1^n \cap S_1$. Hence there exists a $\lambda_i \in S_1$ such that for all n, $\lambda_i = \lambda_{i1} \lambda_{i2} \cdots \lambda_{in}$. By taking norms it follows that $|\lambda_+| \geq |\lambda_-|^n$ for all n, or since $\log |\lambda_-| > 0$ that $n \leq \log |\lambda_+| / \log |\lambda_-|$, a contradiction. Thus $\mathfrak{N}$ is nilpotent. If n_0 is the largest integer such that $S_{n_0} \neq \varnothing$ then the argument above shows that index of nilpotency $\leq n_0 \leq \log |\lambda_+| / \log |\lambda_-|$.

It is worthwhile remarking that the converse to our theorem does hold if the index n of nilpotence is 2 (or 1), i.e., in this case there exists an automorphism of the type discussed above. Of course, if $n = 1$ then $\mathfrak{N}$ is abelian and each $\alpha \in Gl(\mathfrak{N})$ is an automorphism. Thus for example $\alpha = tI$ for $|t| > 1$ satisfies the condition. Now let $n = 2$. Since $\mathfrak{N}$ is nilpotent its center $\mathfrak{Z}(\mathfrak{N}) \neq 0$. Let $\{x_1, \cdots, x_k\}$ be a basis of $\mathfrak{Z}(\mathfrak{N})$ and extend this to a basis $\{x_1, \cdots, x_k, y_1, \cdots, y_m\}$ of $\mathfrak{N}$. Since $[\mathfrak{N}, \mathfrak{N}] \subseteq \mathfrak{Z}(\mathfrak{N})$ we have $[x_i, x_j] = 0$, $i, j = 1, \cdots, k$, $[x_i, y_j] = 0$, $i = 1, \cdots, k$, $j = 1, \cdots, m$ and $[y_i, y_j] \in \mathfrak{Z}(\mathfrak{N})$, $i, j = 1, \cdots, m$. For $t \neq 0$ let $\alpha_t(x_i) = t^2 x_i$, $i = 1, \cdots, k$ and $\alpha_t(y_j) = t y_j$, $j = 1, \cdots, m$. Then α_t is diagonalizable, has for eigenvalues t and t^2 and is in $Gl(\mathfrak{N})$. To see that $\alpha_t \in \mathfrak{A}(\mathfrak{N})$, that is: $\alpha_t [x, y] = [\alpha_t(x), \alpha_t(y)]$ for all $x, y \in \mathfrak{N}$, it suffices to verify this for x and y basis elements. If either x or y equals x_i then $[x, y] = 0$ and hence $\alpha_t [x, y] = 0$. On the other hand, $\alpha_t(x)$ or $\alpha_t(y) \in \mathfrak{Z}(\mathfrak{N})$ therefore $[\alpha_t(x), \alpha_t(y)] = 0$. We may therefore assume $x = y_i$, $y = y_j$ then $[\alpha_t(y_i), \alpha_t(y_j)] = [t y_i, t y_j] = t^2 [y_i, y_j]$ but $\alpha_t [y_i, y_j] = t^2 [y_i, y_j]$ since $[y_i, y_j] \in \mathfrak{Z}(\mathfrak{N})$. An extension of these remarks does not seem possible for higher n's.

Finally, this shows that if G is a connected, simply connected Lie group which is nilpotent of class 2 then G possesses a " contracting family " of automorphisms of the type considered by Koranyi and Vagi in [6], i.e., a 1–parameter group $t \mapsto \beta_t$, a multiplicative homomorphism from the positive reals to the automorphism group $\mathfrak{A}(G)$ of G such that for each $g \in G$, $\lim_{t\to 0} \beta_t(g) = 1$. Since for a simply connected group for each $\alpha_t \in \mathfrak{A}(\mathfrak{G})$ (where $\mathfrak{G}$ is the Lie algebra of G) there exists a unique $\beta_t \in \mathfrak{A}(G)$ such that $\beta_t^{\cdot} = \alpha_t$, the fact that $t \mapsto \alpha_t$ is multiplicative and $\cdot$ is a covariant functor implies that $t \mapsto \beta_t$ is multiplicative. (For facts concerning $\exp_G$ and the relations between $\mathfrak{G}$ and G see G. Hochschild [4]). In the case of a simply connected nilpotent group the map $\exp_G$ is a global diffeomorphism of $\mathfrak{G}$ onto G. Hence for $g \in G$ choose $x \in \mathfrak{G}$ so that $\exp_G(x) = g$, then $\lim_{t\to 0} \beta_t(g) = \lim_{t\to 0} \exp_G(\alpha_t(x)) = \exp_G(\lim_{t\to 0} \alpha_t(x))$. Since $\lim_{t\to 0} \alpha_t(x) = 0$ and $\exp_G$ is a diffeomorphism, $\exp_G(\lim_{t\to 0} \alpha_t(x)) = 1$.

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