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Holder-continuous heat potentials

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Analisi matematica. — *Hölder–continuous heat potentials.* Nota (*) di JOSEF KRÁL, presentata dal Corrisp. G. FICHERA.

RIASSUNTO. — Si studiano i potenziali del calore u in R^{m+1} che soddisfano le condizioni:

$$(O) \qquad |u(x)-u(y)| = O\left(|x_{m+1}-y_{m+1}|^{\frac{1}{2}\alpha} + \sum_{i=1}^{m} |x_i - y_i|^{\alpha}\right),$$

$$(o) \qquad |u(x)-u(y)| = o\left(|x_{m+1}-y_{m+1}|^{\frac{1}{2}\alpha} + \sum_{i=1}^{m} |x_i - y_i|^{\alpha}\right)$$

per $|x-y| \to 0+$. Sono formulate le condizioni sul compatto K sotto le quali si può trovare la misura di supporto K il cui potenziale soddisfa sia (O) che (o). Seguono da ciò le caratterizzazioni degli insiemi delle singolarità delle soluzioni dell'equazione del calore.

With every Borel measure $\mu \geq 0$ with compact support in R^{m+1} we shall associate the heat potential $G\mu$ corresponding to the kernel G defined by

$$G(x) = x_{m+1}^{-\frac{1}{2}m} \exp\left(-\sum_{i=1}^{m} x_i^2/4x_{m+1}\right)$$

for $x = [x_1, \cdots, x_{m+1}]$ with $x_{m+1} > 0$, $G(x) = 0$ for $x_{m+1} \leq 0$.

We are interested in conditions on μ guaranteeing

$$(1) \qquad |G\mu(x) - G\mu(y)| = O\left(|x_{m+1}-y_{m+1}|^{\frac{1}{2}\alpha} + \sum_{i=1}^{m} |x_i - y_i|^{\alpha}\right)$$

as $|x-y| \to 0+$, where α is a fixed exponent satisfying $0 < \alpha < 1$. We shall use H as a generic notation for parallelepipeds of the form

$$(2) \qquad H = K \times I,$$

where K is a cube in R^m of side-length s and I is a one-dimensional interval of length s^2. Such parallelepipeds H in R^{m+1} will be termed distinguished and $s(H) = s$ will denote the side-length of the corresponding cube K. In view of the mixed homogeneity of the heat equation (see (1)) distinguished parallelepipeds replace naturally cubes or balls occurring in connection with

(*) Pervenuta all'Accademia il 27 luglio 1971.

(1) B. FRANK JONES, Jr., *Lipschitz spaces and the heat equation*, « Journal of Math. and Mech. », *18*, 379–409 (1968).

Newton's or Riesz's potentials and permit one to establish the following result (compare [2], [3]):

PROPOSITION 1. *Let* $\mu \geq 0$ *be a Borel measure with compact support. Its heat potential* $G\mu$ *satisfies* (1) *if and only if the following estimate holds for distinguished parallelepipeds* H

(3) $$\mu(H) = O(s^{m+\alpha}(H)) \quad as \quad s(H) \to 0+ .$$

We are now going to characterize those compact sets which are supports of non-trivial measures μ satisfying (3). For this purpose we introduce for any $\beta \geq 0$ the anisotropic measure $\mathfrak{I}^{\beta}$ of the Hausdorff type defining for $M \subset R^{m+1}$

$$\mathfrak{I}^{\beta} M = \lim_{\varepsilon \to 0+} \inf \sum_{n} s^{\beta}(H_n) ,$$

where the infimum is taken over all sequences $\{H_n\}$ of distinguished parallelepipeds such that $\bigcup_n H_n \supset M$ and $s(H_n) \leq \varepsilon$ for all n. Employing a classical method of O. Frostman we obtain.

PROPOSITION 2. *A compact* $K \subset R^{m+1}$ *is a support of a non-trivial measure* μ *satisfying* (3) *for distinguished parallelepipeds* H *if and only if* $\mathfrak{I}^{m+\alpha} K > 0$.

These results show that compact sets $K \subset R^{m+1}$ with $\mathfrak{I}^{m+\alpha} K > 0$ are not removable for solutions u of the heat equation on $R^{m+1} \setminus K$ satisfying the condition (O). If $\mathfrak{I}^{m+\alpha} K < \infty$ then such solutions of the heat equation on the complement of K admit representation described in the following Theorem (whose analogue for harmonic functions is due to Je. P. Dolženko):

THEOREM 1. *Let* $U \subset R^{m+1}$ *be open and suppose that* $K \subset U$ *is compact with* $\mathfrak{I}^{m+\alpha} K < \infty$. *If* u *is a solution of the heat equation on* $U \setminus K$ *satisfying* (O) *then there is a bounded Baire function* φ_u *vanishing on* $R^{m+1} \setminus K$ *such that the measure* $\mu_u = \varphi_u \mathfrak{I}^{m+\alpha}$ *permits to represent* u *in the form*

$$u = G\mu_u + v ,$$

where v *is a solution of the heat equation on the whole of* U.

In particular, if $\mathfrak{I}^{m+\alpha} K = 0$, then $\mu_u = 0$ and K is removable for solutions u of the heat equation on $U \setminus K$ satisfying (O). Combining this with the above propositions we arrive at the following analogue of L. Carleson's result on removable singularities of harmonic functions.

COROLLARY. *A compact* $K \subset U$ *is removable for solutions* u *of the heat equation on* $U \setminus K$ *satisfying* (O) *if and only if* $\mathfrak{I}^{m+\alpha} K = 0$.

(2) L. CARLESON, *Selected problems on exceptional sets*, Van Nostrand 1967.

(3) H. WALLIN, *Existence and properties of Riesz potentials satisfying Lipschitz conditions*, «Mathematica Scandinavica», *19*, 151–160 (1966).

If $\mathcal{H}^{\beta}$ denotes the ordinary β–dimensional Hausdorff measure (whose definition differs from that of $\mathfrak{I}^{\beta}$ in the use of cubes instead of distinguished parallelepipeds for the coverings) then $\mathfrak{I}^{\beta} \leq \mathcal{H}^{\beta-1}$ whenever $\beta \geq 1$, so that $\mathcal{H}^{m-1+\alpha} K = 0$ is sufficient for the removability of K for solutions u of the heat equation considered in the above corollary. (Compare this with a general result of R. Harvey and J. Polking (4) dealing with functions satisfying the Hölder condition with exponent α in all variables).

Consider now non-trivial heat potentials u satisfying (o). It appears that supports of the corresponding measures must have non-σ–finite $\mathfrak{I}^{m+\alpha}$–measure. Combining this with methods of A. S. Besicovitch one can prove the following.

THEOREM 2. *A compact* $K \subset U$ *is removable for solutions* u *of the heat equation on* $U \setminus K$ *satisfying* (o) *if and only if* K *has* σ*–finite* $\mathfrak{I}^{m+\alpha}$*–measure.*

Let us note that an analogous result with $\mathfrak{I}^{m+\alpha}$ replaced by $\mathcal{H}^{m+\alpha}$ holds for harmonic functions h in R^{m+1} satisfying

$$|h(x) - h(y)| = o(|x - y|^{\alpha}) \quad \text{as} \quad |x - y| \to 0+.$$

Detailed proofs together with further references will be given elsewhere.

(4) R. HARVEY and J. POLKING, *Removable singularities of solutions of linear partial differential equations*, « Acta mathematica », *125*, 39–56 (1970).