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Cubical polyhedra and homotopy, II

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Topologia algebrica. — *Cubical polyhedra and homotopy, II.* Nota di JÓZEF BLASS e WŁODZIMIERZ HOLSZTYŃSKI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si fa seguito ad una precedente Nota lincea [1], mostrando l'invarianza omotopica della nozione (qui introdotta) di contigua equivalenza fra morfismi cubici.

The category QP of cubical polyhedra was defined in [1]. In this paper we define the contiguous equivalence relation for cubical morphisms. We obtain a projection category CQ and the projection functor $Cg : QP \to CQ$. This gives an analogy to the homotopy category Ht Top and the homotopy functor $Ht : Top \to Ht\,Top$. The goal of this paper is to prove homotopy invariance of the contiguous equivalence for compact spaces in the following sense. There is a functor $\widetilde{Cg} : Ht\,C \to CQ$ from the homotopy category of compact pairs into CQ, such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\ Ht\,|\,C\ } & Ht\,C \\ {\scriptstyle Q\,|\,C}\downarrow & & \downarrow{\scriptstyle \widetilde{Cg}} \\ QP & \xrightarrow[\ Cg\ ]{} & CQ \end{array}$$

[C is the category of compact pairs, $Q : Top \to QP$ is the cubical functor (see [1]).

The above construction, as we will show in the forthcoming paper, gives a continuous extension of the homotopy invariant functors from the category of finite polyhedra into the category of compact spaces. In particular, we will present purely cubical construction of the Cech homology. For basic definitions used is this paper see [1].

§ 1. CONTIGUOUS CATEGORY

We present a cubical version of acyclic carriers

(1.1) DEFINITION: We say that a family of cubical morphisms

$$G = \{g_t : (V, V_0) \to (W, W_0) : t \in T\}$$

admits a common cubical carrier iff for every face $F_\beta \subset V$ there is a face $F_{\beta'} \subset W$ such that

a) $$\bigcup_{t \in T} f_{g_t}(F_\beta) \subset F_{\beta'}$$

(*) Nella seduta del 18 giugno 1971.

and

b) $F_{\beta'} \subset W_0$ whenever $F_\beta \subset V_0$.

Such a carrier exists iff one can put $F_{\beta'} = \text{carr} \bigcup_{t \in T} f_{q_t}(F_\beta)$.

(1.2) DEFINITION: Cubical morphisms $p, q : (V, V_0) \to (W, W_0)$ of finite polyhedra are said to be *contiguous* iff there exists a finite sequence of morphisms $p = q_0, q_1, \cdots, q_n = q : (V, V_0) \to (W, W_0)$ such that every pair q_{i-1}, q_i admits a common cubical carrier for $i = 1, \cdots, n$.

(1.2′) DEFINITION: Cubical morphisms $p, q : (V, V_0) \to (W, W_0)$, where $V \subset I^A$ and $W \subset I^{A_1}$ are said to be *contiguous* iff for every finite $A'_1 \subset A_1$ there exists a finite $A' \subset A$ such that $A' \supset p(A'_1) \cup q(A'_1)$ and such that the induced morphisms

$$p', q' : p^A_{A'}(V, V_0) \to p^{A_1}_{A'_1}(W, W_0)$$

are contiguous (in the sense of the def. (1.2)).

(1.3) *Remark*: For finite cubical polyhedra both definitions (1.2) and (1.2′) are equivalent.

The following two assertions are obvious.

(1.4) PROPOSITION: The contiguity relation is an *equivalence relation*.

(1.5) PROPOSITION: If $p', p'' : (U, U_0) \to (V, V_0)$ and $q', q'' : (V, V_0) \to (W, W_0)$ are pairs of contiguous morphisms then $q' \circ p'$ and $q'' \circ p''$ are contiguous morphisms. Thus *cubical pairs and contiguous classes of cubical morphisms form a category* CQ.

$Cg : Qp \to CQ$ denotes the canonical projection functor. $Cg(W, W_0) = (W, W_0)$ for every polyhedral pair (W, W_0), and $Cg(q)$ is the contiguous class of q for every cubical morphism q.

§ 2. SWELLING OPERATION

(2.1) DEFINITION: Let $\alpha : A \to A_1$ be a function and $W \subset I^{A_1}$ be a cubical polyhedron. We define a *polyhedron* $W^\alpha \subset I^A$ as follows

$$W^\alpha = \cup \{F_{\tilde\beta} : F_\beta \subset W\}$$

where for β defined on B, $\tilde\beta$ is defined on $\alpha^{-1}(B)$ by $\tilde\beta(a) = \beta \cdot \alpha(a)$.

(2.2) PROPOSITION: $\alpha : A \to A_1$ *is a cubical morphism of* (W, W_0) *into* $(W, W_0)^\alpha$ *for any* $W_0 \subset W \subset I^A$.

In the case of surjective α, W^α is said to be a *swollen polyhedron*, $(W, W_0)^\alpha$ a *swollen pair* and $Cg(\alpha) = (W, W_0) \to (W, W_0)^\alpha$ the *swelling operation*.

Let us notice that for $B \subset A$ and β defined on B the following holds.

(2.3) PROPOSITION: $F_\beta \subset W^\alpha$ iff

$$f_\alpha(W) \cap \bigcap \{G_{(a,\varepsilon)} : \beta(a) = \varepsilon \quad \text{or} \quad a \in A \setminus B\} \neq \varnothing .$$

(2.4) THEOREM: *The swelling operation* $Cg(\alpha) : (W, W_0) \to (W, W_0)^\alpha$ *is an isomorphism (in the category* CQ) *for any cubical pair* (W, W_0).

Proof: Let $\alpha' : A_1 \to A$ be a function such that $\alpha \cdot \alpha' = \mathrm{Id}_{A_1}$, α' is a morphism of W^α into W.

Indeed every face of W^α is contained in a face of the form $F_{\tilde\beta}$, where F_β is a face of W (β is defined on $B \subset A_1$). $\operatorname{carr} \alpha'(F_{\tilde\beta}) = F_\gamma$, where γ is defined on $\alpha'^{-1}(\alpha^{-1}(B)) = (\alpha \cdot \alpha')^{-1}(B) = B$ as follows (see [1]):

$$\gamma = \tilde\beta \cdot (\alpha' | B) = \beta \cdot \alpha \cdot (\alpha' | B) = \beta .$$

Thus image of any face of W^α under α' is contained in a face of W (i.e. α' is a morphism of W^α into W). Therefore it makes a sense to write

$$\alpha' \circ \alpha = \mathrm{Id}_W . \tag{2.5}$$

We will show now that $\alpha \circ \alpha'$ and Id_{W^α} admit a common cubical carrier. It suffices to show that $f_{\alpha \circ \alpha'}(F_{\tilde\beta})$ and $F_{\tilde\beta}$ are contained in a face of W^α for every face $F_\beta \subset W$. But

$$f_{\alpha \circ \alpha'}(F_{\tilde\beta}) = (f_\alpha \cdot f_{\alpha'}) (\operatorname{carr} f_\alpha (F_\beta)) \subset \operatorname{carr} (f_\alpha \cdot f_{\alpha'} \cdot f_\alpha)(F_\beta) = F_{\tilde\beta} .$$

This and (2.5) prove the theorem in the absolute case. The extension to the relative case is trivial.

The following factorization theorem is easy to prove.

(2.6) THEOREM: *Let* $q : (W, W_0) \to (V, V_0)$ *be a cubical morphism of cubical polyhedra* $W_0 \subset W \subset I^A$ *into* $V_0 \subset V \subset I^{A_1}$. *Then*

$$[q : (W, W_0) \to (V, V_0)] =$$
$$= [(\mathrm{Id}_{A_1} : (W, W_0)^q \to (V, V_0)] \circ [q : (W, W_0) \to (W, W_0)^q] .$$

In particular, the above theorem asserts $\mathrm{Id}_{A_1} : (W, W_0)^q \to (V, V_0)$ is a cubical morphism.

§ 3. HOMOTOPICAL INVARIANCE OF CONTIGUITY

In this section we will show *homotopical invariance* of contiguous category of compact pairs.

More precisely, we will show that there exists a functor $\widetilde{Cg} : \mathrm{HtC} \to \mathrm{CQ}$ such that $Cg \circ Q | C = \widetilde{Cg} \circ \mathrm{Ht} | C$ (C and HtC denote categories of compact pairs with continuous map and homotopy classes of continuous maps respectively).

Let (X, X_0) be a topological pair and let A be an abstract set. A map $\alpha : A \rightarrow A(X)$ defines a polyhedron $N\alpha(X_0) \overset{df}{=} (N_{X_0} \alpha(A))^{\alpha}$. We put $N\alpha(X, X_0) = (N\alpha(X), N\alpha(X_0))$. Let $F_\beta \subset I^A$ be a face of I^A for a β defined on a $B \subset A$. We define:

$$\alpha - \operatorname{supp} F_\beta = \cap \{\pi_\varepsilon(\alpha(a)) : \beta(a) \neq -\varepsilon \quad \text{or } a \in A \setminus B\}. \tag{3.1}$$

$$\alpha - \operatorname{supp}_{X_0} F_\beta = X_0 \cap \alpha - \operatorname{supp} F_\beta. \tag{3.2}$$

It is easy to see that

$$F_\beta \subset N\alpha(X_0) \quad \text{iff} \quad \alpha - \operatorname{supp}_{X_0} F_\beta \neq \varnothing. \tag{3.3}$$

(3.4) LEMMA: *Given a topological pair* (X, X_0) *and a pair of abstract sets* A, A_0. *Let* $i : A_0 \rightarrow A$, $q : A \rightarrow A_0$ *and* $\alpha : A \rightarrow A(X)$ *be functions such that*:

a) $q \cdot i = \mathrm{Id}_{A_0}$

b) $\pi_\varepsilon[\alpha \cdot i \cdot q(a)] \subset \pi_\varepsilon[\alpha(a)]$ for all $a \in A$.

Let $\alpha_0 = \alpha \cdot i$, *then*

i) *i is a cubical morphism of* $N\alpha(X, X_0)$ *into* $N\alpha_0(X, X_0)$;

ii) *q is a cubical morphism of* $N\alpha_0(X, X_0)$ *into* $N\alpha(X, X_0)$;

iii) $Cg(q)$ *and* $Cg(i)$ *are inverse isomorphisms in* CQ.

Proof:

i) is obvious.

ii) Let F_{β_0} be a face of $N\alpha_0(X_0)$, where β_0 is defined on $B_0 \subset A_0$. Then $\operatorname{carr} f_q(F_{\beta_0}) = F_\beta$, where β is defined on $B = q^{-1}(B_0)$ by $\beta = \beta_0 \cdot (q \mid B)$. To prove (*ii*) we have to show $F_\beta \subset N\alpha(X_0)$ and this is equivalent to $\alpha - \operatorname{supp}_{X_0} F_\beta \neq \varnothing$ (See (3.3)).

$$\alpha - \operatorname{supp}_{X_0} F_\beta = X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : \beta(a) \neq -\varepsilon \quad \text{or} \quad a \in A \setminus B\}.$$

By (*b*)

$$\alpha - \operatorname{supp}_{X_0} F_\beta \supset X_0 \cap \bigcap \{\pi_\varepsilon[\alpha \cdot i \cdot q(a)] : \beta(a) \neq -\varepsilon \quad \text{or} \quad a \in A \setminus B\}$$
$$= X_0 \cap \bigcap \{\pi_\varepsilon[\alpha_0(q(a))] : \beta(a) \neq -\varepsilon \quad \text{or} \quad a \in A \setminus B\}.$$

Notice that $\beta(a) \neq -\varepsilon$ iff $(\beta_0 \cdot q)(a) \neq -\varepsilon$, and $q : A \rightarrow A_0$ is onto, thus

$$\alpha - \operatorname{supp}_{X_0} F_\beta \supset X_0 \cap \bigcap \{\pi_\varepsilon(\alpha_0(a)) : \beta_0(a) \neq -\varepsilon \quad \text{or} \quad a \in A_0 \setminus B_0\} =$$
$$= \alpha_0 - \operatorname{supp}_{X_0} F_{\beta_0} \neq \varnothing.$$

iii) By (*a*) $Cg(i) \circ Cg(q) = \mathrm{Id}_{N\alpha_0(X, X_0)}$.

Let F_β be a face of $N\alpha(X_0)$, where β is defined on $B \subset A$. $\operatorname{carr} f_{q \circ i}(F_\beta) = \operatorname{carr} F_{\bar\beta}$, where $\bar\beta$ is defined on $\bar B = (q \circ i)^{-1}(B) = (i \cdot q)^{-1}(B)$ by $\bar\beta = \beta \cdot i \cdot q \mid \bar B$.

The set $K = \alpha - \mathrm{supp}_{X_0} F_\beta \cap \alpha - \mathrm{supp}_{X_0} F_{\bar\beta}$ is the α — support of a face of I^A, which contains both F_β and $F_{\bar\beta}$. To show our statement it suffices to prove $K \neq \varnothing$.

$$K = X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : \beta(a) \neq -\varepsilon\} \cap \bigcap \{\pi_\varepsilon(\alpha(a)) ; \tilde\beta(a) \neq -\varepsilon\}.$$

By (*b*)

$$K \supset$$

$$= X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : \beta(a) \neq -\varepsilon\} \cap \bigcap \{\pi_\varepsilon[\alpha \cdot i \cdot q(a)] : \beta \cdot i \cdot q(a) \neq -\varepsilon\} =$$

$$= X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : \beta(a) \neq -\varepsilon\} = \alpha - \mathrm{supp}_{X_0} F_\beta \neq \varnothing .$$

Thus $\mathrm{Id}_{N\alpha(X,X_0)}$ and $q \circ i$ admit a common carrier, i.e.

$$Cg(q) \circ Cg(i) = \mathrm{Id}_{N\alpha(X,X_0)} .$$

We define $(L, M) \times I = (L \times I, M \times I)$. Let $i_t : (X, X_0) \to (X, X_0) \times I$ be the canonical embedding $i_t(x, x_0) = [(x, t), (x_0, t)]$. For $A_0 \subset A(X)$ we define $A_0 \times I = \{G \times I : G \in A_0\} \subset A(X \times I)$.

(3.5) LEMMA: *Given* $A \subset A(X \times I)$ *and* $A_0 \times A(X)$. *Let*

$$j : A_0 \to A \cup (A_0 \times I) \qquad \textit{and} \quad q : A \cup (A_0 \times I) \to A_0$$

be functions such that $j(G) = G \times I$, $q \cdot j = \mathrm{Id}_{A_0}$ *and* $\pi_\varepsilon(j \cdot q(G)) \subset \pi_\varepsilon(G)$ *for every* $G \in A$. *Let* $A^1 = i_{-1}^{-1}(A) \cup i_1^{-1}(A) \cup A_0$. *Then* $N_A^{A^1}(i_{-1})$ *and* $N_A^{A^1}(i_1)$ *are contiguous morphisms of* $NA^1(X, X_0)$ *into* $NA((X, X_0) \times I)$.

Proof:

$$N_A^{A^1}(i_\varepsilon) = i_A^{A \cup (A_0 \times I)} \circ N_{A \cup (A_0 \times I)}^{A^1}(i_\varepsilon) \ ^{(1)} .$$

Thus it suffices to show that $N_{A \cup (A_0 \times I)}^{A^1}(i_{-1})$ and $N_{A \cup (A_0 \times I)}^{A^1}(i_1)$ are contiguous.

$$(3.6) \qquad N_{A \cup (A_0 \times I)}^{A^1}(i_\varepsilon) = N_{A \cup (A_0 \times I)}^{i_\varepsilon^{-1}(A) \cup A_0}(i_\varepsilon) \circ i_{i_\varepsilon}^{A^1} - 1(A) \cup A_0 .$$

Let us notice that the cubical morphism

$$j : N(A \cup (A_0 \times I))(X, X_0) \to NA_0(X, X_0)$$

is induced by the canonical projection $P : (X, X_0) \times I \to (X, X_0)$, i.e.

$$(3.7) \qquad i_{A_0}^{i_\varepsilon^{-1}(A) \cup A_0} = j \circ N_{A \cup (A_0 \times I)}^{i_\varepsilon^{-1}(A) \cup A_0}(i_\varepsilon) .$$

(1) For $C \subset D$, $i_C^D = C \to D$ denotes the *inclusion map*. In the above such inclusion maps are cubical morphisms.

By lemma (3.5) $Cg(j)$ is an isomorphism. Thus, by (3.6) and (3.7)

$$Cg\,(N^{A^1}_{A\cup(A_0\times I)}(i_\varepsilon)) = Cg\,(j)^{-1}\circ Cg\,(i_{A_0}^{\overset{-1}{i_\varepsilon}(A)\cup A_0})\circ$$

$$\circ\, Cg\,(i^{A^1}_{\overset{-1}{i_\varepsilon}(A)\cup A_0}) = Cg\,(j)^{-1}\circ Cg\,(i^{A^1}_{A_0})$$

does not depend on ε.

(3.8) THEOREM: *The embeddings* $i_{-1}, i_1 : (X, X_0)\to(X, X_0)\times I$ *induce contiguous morphisms* $Q(i_{-1})$ *and* $Q(i_1) : Q(X, X_0)\to Q((X, X_0)\times I)$ *for every compact pair* (X, X_0).

Proof: Let $A\in \mathrm{Fin}\,A(X\times I)$. There exists a finite sequence of real numbers $-1 = t_0 < t_1 < \cdots < t_n = 1$, a sequence $A_1, \cdots, A_n \in \mathrm{Fin}\,A(X)$ and a sequence of functions $p_k : A\to A_k$ for $k = 1, \cdots, n$, such that $\pi_\varepsilon(p_k(G)\times[t_{k-1}; t_k])\subset\pi_\varepsilon(G)$.

Put $A^1_k = \overset{-1}{i}_{t_{k-1}}(A)\cup\overset{-1}{i}_{t_k}(A)\cup A_k$.

By lemma (3.5), $N^{A^1_k}_A(i_{t_{k-1}})$ and $N^{A^1_k}_A(i_{t_k})$ are contiguous. Thus for $A^1 = \bigcup_{k=1}^{n} A^1_k$,

$$N^{A^1}_A(i_{t_{k-1}}) \quad\text{and}\quad N^{A^1}_A(i_{t_k})$$

are contiguous for $k = 1, \cdots, n$. Consequently $N^{A^1}_A(i_{-1})$ and $N^{A^1}_A(i_1)$ are contiguous. Thus, by definition (1.2′), $Q(i_{-1})$ and $Q(i_1)$ are contiguous.

(3.9) COROLLARY: *If* f *and* $g : (X, X_0)\to(Y, Y_0)$ *are homotopically equivalent and* (X, X_0) *is a compact pair, then* Qf *and* $Qg : Q(X, X_0)\to Q(Y, Y_0)$ *are contiguous.*

Remark, that (Y, Y_0) is an arbitrary topological pair.

BIBLIOGRAPHY

[1] J. BLASS and W. HOLSZTYŃSKI, *Cubical polyhedra and homotopy, I*, « Rend. Acc. Naz. Lincei », *50*, 131–138 (1971).