Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

H. D. Pande

Lie derivation in a Minkowskian Finsler space

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.6, p. 699–702.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_50_6_699_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1971.

Geometria differenziale. — *Lie derivation in a Minkowskian Finsler space.* Nota di H. D. PANDE, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — La derivata Lie fu introdotta per prima da Van Danzig [2] (1), ed E. Cartan [1] considerava il vettore di deformazione come funzione di sola posizione. Nel 1939, Davies [3] enunciò la teoria della deformazione infinitesimale negli spazi metrici generalizzati, prendendo il vettore lungo il quale la deformazione viene considerata come funzione sia di posizione che di direzione. Abbiamo studiato qui le derivate di Lie in uno spazio minkowskiano e la forma generalizzata degli operatori che sono stati usati da Schouten [8] nel suo trattamento dei problemi di deformazione nello spazio riemanniano.

1. *Infinitesimal deformation and the Lie derivative.* Let us consider a contravariant vector field $X^i(x^k, \xi^k)$ depending on positional coordinates and direction field $\xi^k = \xi^k(x)$. Its covariant derivative at the point x^k in the direction of $\xi^k(x)$ is given by

$$(1.1) \qquad X^i_{;k}(x, \xi) = (\partial_k X^i) + \left(\frac{\partial X^i}{\partial \xi^l}\right)\left(\frac{\partial \xi^l}{\partial x^k}\right) + P^{*i}_{\pi k} X^\pi ,$$

where the functions $P^{*i}_{\pi k}(x, \xi)$ are symmetric in their lower indices.

We consider the infinitesimal transformation

$$(1.2) \qquad \bar{x}^i = x^i + u^i(x, \xi)\, dT ,$$

where $u^i(x, \xi)$ is a vector field depending on position coordinates x^k and directional field $\xi^i(x^j)$ defined over the region of the space considered and dT is to be treated as an infinitesimal constant. The points therefore undergo a small displacement of amount $dx^i = u^i(x, \xi)\, dT$.

Differentiating (1.2) with respect to x^j we obtain

$$(1.3) \qquad \partial_j \bar{x}^i = \delta^i_j + (\partial_j u^i)\, dT , \qquad \partial_j \equiv \partial/\partial x^j .$$

The corresponding variations in the direction $\xi^i(x^k)$ are given by

$$(1.4) \qquad \bar{\xi}^i = \xi^i + (\partial_j u^i)\, \xi^j\, dT + \left(\frac{\partial u^i}{\partial \xi^j}\right) d\xi^j\, dT .$$

The value taken on by the vector field $X^i(x, \xi)$ at $(\bar{x}, \bar{\xi})$ will be given to first order of dT by

$$(1.5) \qquad X^i(\bar{x}, \bar{\xi}) = X^i(x, \xi) + (\partial_k X^i)\, u^k\, dT + \frac{\partial X^i}{\partial \xi^h}\left\{(\partial_j u^h)\, \xi^j + \left(\frac{\partial u^h}{\partial \xi^j}\right) d\xi^j\right\} dT .$$

(*) Nella seduta del 17 aprile 1971.

(1) Numbers in brackets refer to the references at the end of the paper.

Using Schouten's notation $\overset{1}{d}X^i$ for the difference $X^i(\bar{x}, \bar{\xi}) - X^i(x, \xi)$, we can write

$$\overset{1}{d}X^i = (\partial_k X^i)\, u^k\, dT + \left(\frac{\partial X^i}{\partial \xi^h}\right) \left\{ (\partial_j u^h)\, \xi^j + \left(\frac{\partial u^h}{\partial \xi^j}\right) d\xi^j \right\} dT. \tag{1.6}$$

If we denote by $\bar{X}^i(\bar{x}, \bar{\xi})$ the transform of the vector field $X^i(x, \xi)$ by the infinitesimal transformation (1.2), we have

$$\bar{X}^i(\bar{x}, \bar{\xi}) = X^j(\partial_j \bar{x}^i) = X^i(x, \xi) + X^j(\partial_j u^i)\, dT. \tag{1.7}$$

The vector $\bar{X}^i(\bar{x}, \bar{\xi})$ is called the displaced vector of $X^i(x, \xi)$ from (x, ξ) to $(\bar{x}, \bar{\xi})$ and we write

$$\overset{2}{d}X^i = X^j(\partial_j u^i)\, dT. \tag{1.8}$$

If we now denote by $X^i(x, \xi \,\|\, \bar{x}, \bar{\xi})$ the vector transported by parallelism from (x, ξ) to $(\bar{x}, \bar{\xi})$, then since

$$DX^i = dX^i + P^{*i}_{\pi j}(x, \xi)\, X^\pi\, dx^j = 0, \tag{1.9}$$

we can write

$$\begin{aligned} \overset{3}{d}X^i &= X^i(x, \xi \,\|\, \bar{x}, \bar{\xi}) - X^i(x, \xi) \\ &= - P^{*i}_{\pi j}(x, \xi)\, X^\pi u^j\, dT. \end{aligned} \tag{1.10}$$

We define

$$\overset{(r,s)}{D} = \frac{\overset{r}{d} - \overset{s}{d}}{dT}; \qquad r, s = 1, 2, 3 \;;\; r \neq s. \tag{1.11}$$

For a contravariant vector field $X^i(x, \xi)$ we obtain three invariant derivatives as below:

$$\begin{aligned} \overset{(1,2)}{D} X^i(x, \xi) = {} & X^i_{;k} u^k + \left(\frac{\partial X^i}{\partial \xi^h}\right) \left\{ \xi^j u^h_{;j} - u^j \xi^h_{;j} + \left(\frac{\partial u^h}{\partial \xi^j}\right) d\xi^j \right\} + \\ & + \left\{ \frac{\partial \xi^l}{\partial x^j} \right\} \left\{ X^j \left(\frac{\partial u^i}{\partial \xi^l}\right) - \xi^j \left(\frac{\partial X^i}{\partial \xi^h}\right) \left(\frac{\partial u^h}{\partial \xi^l}\right) + \left(\frac{\partial X^i}{\partial \xi^l}\right) u^j \right\} - X^j u^i_{;j}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} \overset{(1,3)}{D} X^i(x, \xi) = {} & X^i_{;k} u^k + \left(\frac{\partial X^i}{\partial \xi^h}\right) \left\{ \xi^j u^h_{;j} - u^k \xi^h_{;k} + \left(\frac{\partial \xi^h}{\partial x^k}\right) u^k - \right. \\ & \left. - \left(\frac{\partial u^h}{\partial \xi^l}\right) \left(\frac{\partial \xi^l}{\partial x^j}\right) \xi^j + \left(\frac{\partial u^h}{\partial \xi^j}\right) d\xi^j \right\}, \end{aligned} \tag{1.13}$$

and

$$\overset{(2,3)}{D} X^i(x, \xi) = - u^i_{;j} X^j + \left(\frac{\partial u^i}{\partial \xi^l}\right) \left(\frac{\partial \xi^l}{\partial x^j}\right) X^j, \tag{1.14}$$

which are the generalizations to Minkowskian Finsler space of the Lie derivative, the covariant derivative and the apparent derivative of Schouten and Van Kampen [8] in ordinary Riemannian space. The Lie derivative of $X^i(x, \xi)$ is obtained by Nisha Rani [6].

The derivation of formulae (1.11) can be extended to an arbitrary tensor $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x, \xi)$ in the following manner:

$$\overset{(1,2)}{D} T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x, \xi) = T^{i_1 \cdots i_r}_{j_1 \cdots j_s ; k} u^k + \left(\partial T^{i_1 \cdots i_r}_{j_1 \cdots j_s} / \partial \xi^h\right) \cdot \tag{1.15}$$

$$\cdot \left\{ \xi^k u^h_{;k} - u^k \xi^h_{;k} + \left(\frac{\partial u^h}{\partial \xi^l}\right) d\xi^l - \left(\frac{\partial u^h}{\partial \xi^k}\right)\left(\frac{\partial \xi^k}{\partial x^m}\right) \xi^m + \left(\frac{\partial \xi^h}{\partial x^m}\right) u^m \right\} -$$

$$- \sum_p T^{i_1 \cdots i_{p-1}, k, i_{p+1}, \cdots, i_r}_{j_1 \cdots j_s} \left\{ u^{i_p}_{;k} - \left(\frac{\partial u^{i_p}}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^k}\right) \right\} +$$

$$+ \sum_q T^{i_1 \cdots i_r}_{j_1 \cdots j_{q-1}, k, j_{q+1}, \cdots, j_s} \left\{ u^k_{;j_q} - \left(\frac{\partial u^k}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^{j_q}}\right) \right\},$$

$$\overset{(1,3)}{D} T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x, \xi) = T^{i_1 \cdots i_r}_{j_1 \cdots j_s ; k} u^k + \left(\partial T^{i_1 \cdots i_r}_{j_1 \cdots j_s} / \partial \xi^h\right) \cdot \tag{1.16}$$

$$\cdot \left\{ \xi^j u^h_{;j} - u^k \xi^h_{;k} + \left(\frac{\partial \xi^h}{\partial x^k}\right) u^k - \left(\frac{\partial u^h}{\partial \xi^l}\right)\left(\frac{\partial \xi^l}{\partial x^j}\right) \xi^j + \left(\frac{\partial u^h}{\partial \xi^j}\right) d\xi^j \right\},$$

and

$$\overset{(2,3)}{D} T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x, \xi) = \sum_p T^{i_1 \cdots i_{p-1}, k, i_{p+1}, \cdots, i_r}_{j_1 \cdots j_s} \left\{ - u^{i_p}_{;k} + \left(\frac{\partial u^{i_p}}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^k}\right) \right\} - \tag{1.17}$$

$$- \sum_q T^{i_1 \cdots i_r}_{j_1 \cdots j_{q-1}, k, j_{q+1}, \cdots, j_s} \left\{ - u^k_{;j_q} + \left(\frac{\partial u^k}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^{j_q}}\right) \right\}.$$

2. *The Lie derivative of the metric tensor.* Let $A^{ij}(x, \xi)$ be the component of a second order tensor. Then $A^{ij}(x, \xi)$ at $(\bar{x}, \bar{\xi})$ and deformed components $\bar{A}^{ij}(\bar{x}, \bar{\xi})$ are given by

$$A^{ij}(\bar{x}, \bar{\xi}) = A^{ij}(x, \xi) + (\partial_k A^{ij}) u^k dT + \tag{2.1}$$

$$+ \left\{ \left(\frac{\partial A^{ij}}{\partial \xi^h}\right)\left(\frac{\partial u^h}{\partial x^j}\right) \xi^j + \left(\frac{\partial A^{ij}}{\partial \xi^h}\right)\left(\frac{\partial u^h}{\partial \xi^j}\right) d\xi^j \right\} dT,$$

$$\bar{A}^{ij}(\bar{x}, \bar{\xi}) = A^{ij}(x, \xi) + \left(A^{ik} \frac{\partial u^j}{\partial x^k} + A^{kj} \frac{\partial u^i}{\partial x^k}\right) dT. \tag{2.2}$$

Therefore the Lie derivative of $A^{ij}(x, \xi)$ can be written in the form

$$\overset{(1,2)}{D} A^{ij}(x, \xi) = A^{ij}_{;k} u^k + (\partial A^{ij} / \partial \xi^h) \cdot \tag{2.3}$$

$$\left\{ \xi^k u^h_{;k} - u^m \xi^h_{;m} + \left(\frac{\partial u^h}{\partial \xi^m}\right) d\xi^m + \left(\frac{\partial \xi^h}{\partial x^m}\right) u^m - \left(\frac{\partial u^h}{\partial \xi^s}\right)\left(\frac{\partial \xi^s}{\partial x^m}\right) \xi^m \right\} -$$

$$- A^{ik} u^j_{;k} - A^{kj} u^i_{;k} + A^{ik} \left(\frac{\partial u^j}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^k}\right) + A^{kj} \left(\frac{\partial u^i}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^k}\right).$$

Similarly, the Lie derivative of a second order covariant tensor $A_{ij}(x, \xi)$ can be obtained easily.

In particular, the Lie derivative of the metric tensor is given by

$$(2.4)\quad \overset{(1,2)}{\mathrm{D}} g_{ij}(x,\xi) = 2\mathrm{C}_{ijh}(x,\xi)\left\{ u^h_{;k}\,\xi^k + \left(\frac{\partial u^h}{\partial \xi^m}\right) \mathrm{d}\xi^m + \left(\frac{\partial \xi^h}{\partial x^m}\right) u^m - \left(\frac{\partial u^h}{\partial \xi^s}\right)\left(\frac{\partial \xi^s}{\partial x^m}\right)\xi^m \right\} +$$

$$+ g_{ik}\, u^k_{;j} + g_{kj}\, u^k_{;i} - g_{ik}\left(\frac{\partial u^k}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^j}\right) - g_{kj}\left(\frac{\partial u^k}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^i}\right).$$

where we have used the generalization of Ricci's lemma $g_{ij\,;\,k} - 2\mathrm{C}_{ijh}\,\xi^h_{;k} = 0$, and $\mathrm{C}_{ijh}(x\,,\xi) \equiv \frac{1}{2}\,\partial g_{ij}(x\,,\xi)/\partial\xi^h$.

References

[1] E. Cartan, *Les espaces de Finsler*, « Actualités Sci et Ind. », Hermann and Cie, Paris, 79 (1934).

[2] D. van V. Dantzig, *Zur allgemeinen projektiven differential geometrie I, II*, « Proc. Kon. Akad., Amsterdam », *35*, 524–534; 535–542 (1932).

[3] E. T. Davies, *Lie derivation in generalized metric spaces*, « Ann. Mat. Pura Appl. », *18*, 261–274 (1939).

[4] R. S. Mishra, *A course in tensors with applications to Riemannian Geometry*, Pothishala Private Limited, Allahabad 1965.

[5] H. D. Pande, *The projective transformation in a Finsler space*, « Atti Accad. Naz. Lincei, Rendiconti », *43* (6), 480–484 (1967).

[6] N. Rani, *Theory of Lie derivatives in a Minkowskian space*, Tensor (N.S.), *20*, 100–102 (1969).

[7] H. Rund, *The differential geometry of Finsler spaces*, Springer–Verlag (1959).

[8] J. A. Schouten and E. R. van Kampen, *Beiträge zur theorie der deformation*, « Prace Mat. Fiz. Warszawa », *41*, 1–19 (1933).

[9] K. Yano, *The theory of Lie derivatives and its applications*, North–Holland Pub. Co., Amsterdam 1955.