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Analisi funzionale. — *An existence theorem for quasimonotone operators.* Nota di BRUCE CALVERT (*) e JEFFREY RONALD LESLIE WEBB (**), presentata (***) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si dà un esempio di un operatore differenziale che non è pseudomonotono, ma per il quale si dimostra un teorema di suriettività per mezzo della teoria astratta degli operatori quasimonotoni introdotti in questo lavoro.

The existence theory for nonlinear partial differential equations has been tackled over the last few years by the theory of monotone operators and their generalizations. One of the most powerful abstract results to date seems to be one recently announced by Browder regarding pseudomonotone operators. This result is obtained as a consequence of a generalized theory of degree which is applicable to the pseudomonotone operators. In the present paper, we introduce a class of mappings, the quasimonotone operators, which includes the pseudomonotone class and prove a similar existence theorem for this class. Although the generalized degree theory is available, we prefer a more direct and simpler approach.

Our approach enables us to give a mapping theorem for operators satisfying condition (S) which was obtained under the hypothesis $(S)_+$ by Browder [4], using a generalised degree theory, and by Hess [5], using finite dimensional approximations; it is not clear that the result for (S) could be obtained from the degree theory. We give a concrete example to show that our theorem is a real extension of the previous one.

We shall work throughout in a reflexive separable Banach space, denoted by X, and we shall suppose that it is endowed with the equivalent norm which makes X and X^* locally uniformly convex. We recall that X is said to be locally uniformly convex if for each x in X with $\|x\| = 1$ and each $\varepsilon > 0$ there exists $\eta > 0$ such that $\|y\| \leq 1$ and $\|x + y\| \geq 2(1 - \eta)$ together imply that $\|x - y\| < \varepsilon$. The existence of the equivalent norm in our case is assured by results of Kadek [6] and Asplund [1].

Let $T : X \to X^*$ be a (not necessarily linear) mapping. T is said to be bounded if the image of each bounded set is bounded. T is said to be pseudomonotone if it is bounded and satisfies:

(PM): Whenever a sequence $\{x_j\}$ in X converges weakly to an element x and $\limsup\,(Tx_j, x_j - x) \leq 0$, then $\liminf\,(Tx_j, x_j - y) \geq (Tx, x - y)$ for all y in X.

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This definition is due to Brézis [2]. Note that we have used the usual notation (f, x) to denote the value of f in X^* at x in X.

A weaker requirement than pseudomonotonicity is condition (M) of Brézis [2]:

(M): If $x_n \to x$ weakly in X and $Tx_n \to f$ weakly in X^* and $\limsup (Tx_n, x_n) \leq (f, x)$ then $Tx = f$.

The conditions (S) and $(S)_+$ were defined by Browder (see e.g.; [4]) and are the following:

(S): $x_n \to x$ weakly in X and $\lim (Tx_n, x_n - x) = 0$ together imply that $x_n \to x$.

$(S)_+$: $x_n \to x$ weakly and $\limsup (Tx_n, x_n - x) \leq 0$ imply that $x_n \to x$.

The quasimonotone operators are those bounded mappings $T: X \to X^*$ which satisfy

(Q): $x_n \to x$ weakly implies that $\limsup (Tx_j, x_j - x) \geq 0$.

We recall that T is said to be demicontinuous if $x_j \to x$ implies that $Tx_j \to Tx$ weakly. If T is pseudomonotone it is automatically demicontinuous.

LEMMA 1: *Let* $T: X \to X^*$ *be bounded, demicontinuous and satisfy* (S) *and suppose there is a constant* $M \geq 0$ *such that* T *is an odd map for* $\|x\| \geq M$, *that is,* $T(-x) = -T(x)$ *whenever* $\|x\| \geq M$. *Then there exists* x *such that* $Tx = 0$.

Proof: By the separability of X, there is a countable family X_j of finite dimensional subspaces of X such that $X_j \subset X_{j+1}$ for all j and with union dense in X. Let ψ_j be the injection map of X_j into X and ψ_j^* its adjoint. Then $T_j = \psi_j^* T \psi_j$ is a continuous map of X_j into X_j^* and is odd for $\|x\| \geq M$. By the finite dimensional Borsuk theorem there exists x_j in X_j with $\|x_j\| \leq M$ such that $T_j x_j = 0$. By the boundedness of $\{x_j\}$ and the reflexivity of X, there is a subsequence $\{x_k\}$, such that $x_k \to x$ weakly in X. Then $(Tx_k, x_k) = (T_k x_k, x_k) = 0$ for all k and further $(Tx_k, v) \to 0$ for all v in $\bigcup_{j=1}^{\infty} X_j$, for if v is in X_n, $(Tx_k, v) = (T_k x_k, v) = 0$ for all $k \geq n$. It follows that $(Tx_k, x_k - v) \to 0$ for all v in X because the convergence holds for v in a dense subset and $\{Tx_k\}$ is bounded. Taking $v = x$ and applying (S) we obtain $x_k \to x$. By demicontinuity $Tx_k \to Tx$ weakly; however, the above established that $Tx_k \to 0$ weakly so that $Tx = 0$.

Remark. The same conclusion holds for operators that satisfy (M) instead of (S); the proof is a routine modification of the above.

The next Theorem is the key to proving results for quasimonotone operators: we prove it for maps satisfying (S) for its independent interest though in the sequel we only use it under the stronger assumption $(S)_+$.

THEOREM 1. *Let* $B = B(0; r)$ *be a closed ball in* X *centred at* 0 *and with radius* $r > 0$. *Let* $F: B \times [0, 1]$ *be such that* $F_t \equiv F(\cdot, t)$ *is a demicontinuous bounded map satisfying* (S) *for each* t *in* $[0, 1]$.

Suppose also that

H1 : F is continuous in t uniformly for x in B, that is, $t_j \to t$ and $\{x_j\} \subset B$ imply $F(x_j, t_j) - F(x_j, t) \to 0$.

H2 : $F(x, t) \neq 0$ for all x in $bdy(B) = \{x : \|x\| = r\}$, and all t in $[0, 1]$.

H3 : F_0 is odd on $bdy(B)$.

Then there exists x in B such that $F_1(x) = 0$.

Proof. Without loss of generality we assume that $r = 1$. Define a map $A : B \to X^*$ by

$$A(x) = \begin{cases} F\left(\frac{x}{\|x\|}, 2(1 - \|x\|)\right) & \text{if } \|x\| \geq \frac{1}{2} \\ F(2x, 1) & \text{if } \|x\| \leq \frac{1}{2}. \end{cases}$$

Clearly A is bounded and it is routine to establish its demicontinuity. We show that it satisfies (S). Suppose that $x_n \to x$ weakly and that $\lim (Ax_n, x_n - x) = 0$. Let $\{x_k\}$ be the subsequence of $\{x_n\}$ for which $\|x_k\| \leq \frac{1}{2}$ and $\{x_j\}$ be such that $\|x_j\| > \frac{1}{2}$. From the definition of A and the fact that F_1 satisfies (S) we see that $x_k \to x$. The sequence $\{x_j\}$ has a subsequence, which we denote again by $\{x_j\}$, such that $\|x_j\| \to \lambda \in \left[\frac{1}{2}, 1\right]$. As a consequence of H 1 we find

$$\lim \left(F\left(\frac{x_j}{\|x_j\|}, 2(1 - \lambda)\right), x_j - x\right) = 0.$$

However, we also have, because F is bounded and $\|x_j\| > \frac{1}{2}$,

$$\lim \left(F\left(\frac{x_j}{\|x_j\|}, 2(1 - \lambda)\right), \frac{x}{\|x_j\|} - \frac{x}{\lambda}\right) = 0.$$

This yields

$$\lim \left(F\left(\frac{x_j}{\|x_j\|}, 2(1 - \lambda)\right), \frac{x_j}{\|x_j\|} - \frac{x}{\lambda}\right) = 0.$$

By the (S) conditions imposed we have $\frac{x_j}{\|x_j\|} \to \frac{x}{\lambda}$, that is, $x_j \to x$ and *a posteriori* $\lambda = \|x\|$. Hence the whole sequence $x_j \to x$ as the above argument can be applied to all subsequences.

It follows that $x_n \to x$ and so A satisfies (S). Since A coincides with F_0 on $bdy(B)$, it is odd there; Lemma 1 applies and furnishes x in B such that $A(x) = 0$. However this implies that $F_1(y) = 0$ for some y in B, for to the contrary we would have $A(x) \neq 0$ for all x in B.

Under the assumption $(S)_+$, this result has been proved by Browder [3] and by Hess [5]. Browder used degree arguments and also took X to be separable but Hess, by using finite dimensional approximation methods, removed the separability assumption. If we had established Lemma I for nonseparable X we could do the same for condition (S).

For the next result we need the notion of a duality map. A map $J : X \to X^*$ is called the duality map if $\|J(x)\| = \|x\|$ and $(J(x), x) = \|x\|^2$ for all x in X. As X^* is locally uniformly convex J is uniquely determined by these requirements (see e.g. [3]). The following Lemma characterizes the quasimonotone operators.

LEMMA 2. *Let* $T : X \to X^*$ *be bounded and demicontinuous. Then* T *is quasimonotone if and only if for all* $\varepsilon > 0$, $T + \varepsilon J$ *is bounded, demicontinuous and satisfies* $(S)_+$.

Proof. Let T be quasimonotone and suppose that $x_j \to x$ weakly and $\limsup ((T + \varepsilon J)(x_j), x_j - x) \leq 0$. From (Q) we obtain $\limsup (\varepsilon J(x_j) - \varepsilon J(x), x_j - x) = \limsup (\varepsilon J(x_j), x_j - x) \leq 0$. Since J is monotone each term is nonnegative so the $\liminf \geq 0$. This implies that $(J(x_j) - J(x), x_j - x) \to 0$ and so $x_j \to x$ (see e.g. [3]).

Conversely, suppose that T does not satisfy (Q), that is, there is a sequence $x_j \to x$ weakly for which $\limsup (T(x_j), x_j - x) = -\delta$, $\delta > 0$. Note that this shows that $x_j \not\to x$. Since $\{x_j\}$ is bounded, say $\|x_j\| \leq M$ we have $|(\varepsilon J(x_j), x_j - x)| \leq \varepsilon 2M^2 \leq \delta/2$ for ε sufficiently small. Then for such ε $\limsup ((T + \varepsilon J)(x_j), x_j - x) \leq -\delta/2 \leq 0$. However, as noted above, $\{x_j\}$ does not converge to x, so that $(S)_+$ does not hold for $T + \varepsilon J$ for all $\varepsilon > 0$.

We can now give the surjectivity Theorem for quasimonotone operators.

THEOREM 2. *Let* T_t, $0 \leq t \leq 1$, *be a family of quasimonotone, demicontinuous operators satisfying* H 1. *Suppose that* T_0 *is an odd map for large* $\|x\|$. *Further, assume there is a continuous function* $\varphi : R^+ \to R^+$ *such that for all* f *in* X^* *and* t *in* $[0, 1]$, $T_t(x) = f$ *implies that* $\|x\| \leq \varphi(\|f\|)$. *Suppose* $T(B)$ *is closed for each closed ball* B. *Then* T *is onto.*

Proof. Let f be an arbitrary point of X^* and choose $r > 0$ such that T_0 is odd for $\|x\| \geq r$ and $\|T_t(x)\| > 1 + \|f\|$ for $\|x\| \geq r$ and for all t in $[0, 1]$; (that this can be done may be seen as follows: suppose not, then there are sequences $\{x_j\}$, $\{t_j\}$ such that $\|x_j\| \to \infty$ and $\|T_{t_j}(x_j)\| \leq 1 + \|f\|$. Write $g_j = T_{t_j}(x_j)$ and apply the hypothesis to obtain $\|x_j\| \leq \varphi(\|g_j\|)$, which contradicts $\|x_j\| \to \infty$ because $\|g_j\| \leq 1 + \|f\|$ and φ is continuous).

Now choose $\varepsilon > 0$ such that $\varepsilon r < 1$ and define

$$F(x, t) = T_t(x) + \varepsilon J(x) - tf, \qquad 0 \leq t \leq 1, \quad \|x\| \leq r.$$

One easily sees that the hypotheses of the preceding Theorem hold so there exist x_ε with $\|x_\varepsilon\| \leq r$ such that $T_1(x_\varepsilon) + \varepsilon J(x_\varepsilon) = f$. Let $\varepsilon \to 0 : T_1(x_\varepsilon) \to f$. By the closedness hypothesis there is an x, $\|x\| \leq r$ such that $T_1(x) = f$. The proof is complete.

COROLLARY. *Let* $T : X \to X^*$ *be quasimonotone and demicontinuous and suppose there are positive constants* λ *and* M *such that* $(T(x), x) \geq -\lambda \|x\|$ *for* $\|x\| \geq M$. *Then* $T + \varepsilon J$ *is surjective for all* $\varepsilon > 0$. *If, in addition,* $T(B)$ *is closed for each closed ball* B *and if* T *is coercive, that is,* $(T(x), x)/\|x\| \to \infty$ *as* $\|x\| \to \infty$, *then* T *is surjective.*

Proof. For f in X^* we apply Theorem 1 to the homotopy $F(x, t) = tT(x) + \varepsilon J(x) - tf$. When T is coercive the solutions x_ε of $T(x_\varepsilon) + \varepsilon J(x_\varepsilon) = f$ are bounded independent of ε; the closedness hypothesis completes the proof.

The sum of two quasimonotone operators is again quasimonotone as we see from Lemma 2. Consequently the sum of a monotone operator with a compact one is quasimonotone but need not be pseudomonotone if the compact map is not completely continuous. We recall that by compact we mean continuous and the image of each bounded set relatively compact; by completely continuous we mean that weakly convergent sequences are mapped into strongly convergent ones. In order to apply our Theorem to the sum of a monotone operator with a compact one we need the closedness hypothesis to hold. This will be so if A is proper; we have not found any examples of second order and zero order operators which are monotone and proper that do not also satisfy $(S)_+$. Therefore we give an operator of first order.

Example: For a function u in $L^2(S^1)$, S^1 the unit circle in R^2 (which may be identified with a periodic function on $[0, 2\pi]$) the Fourier series expansion

$$u(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, \qquad 0 \leq x \leq 2\pi,$$

is valid, where $a_n = (2\pi)^{-1} \int_0^{2\pi} u(x) e^{-inx} dx$, as $(2\pi)^{-1/2} e^{inx}$ is a complete orthonormal basis. For a function u in $C^\infty(S^1)$, $\|u\|^2 = \sum_{n=-\infty}^{\infty} |na_n^2|$ defines a seminorm, and $\|u\| = 0$ implies that $u = a_0$, a constant function. We define H to be the completion of $C^\infty(S^1)/\{\text{constants}\}$ with respect to this norm. We define H^* to be the completion of those functions f in $C^\infty(S^1)$ which are perpendicular to all constants (in the L^2 sense), that is, $f = \sum_{n=-\infty}^{\infty} b_n e^{inx}$, $b_0 = 0$, with respect to the norm $\|f\| = \sum_{n=-\infty}^{\infty} |n^{-1} b_n^2|$. A natural pairing between elements in H and H^* is provided by the L^2 pairing (or equivalently l^2), namely for $\tilde{u} = u + \{\text{constants}\}$ in H and f in H^*, let

$$(f, \tilde{u}) = \int_0^{2\pi} u(x) \bar{f}(x) dx = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n,$$

a_n, b_n the respective Fourier coefficients.

One may check that this is a well defined continuous bilinear form on $H \times H^*$. In fact H^* is the dual space of H under this pairing.

For $\tilde{u} = u + \{\text{constants}\}$ and $\tilde{v} = v + \{\text{constants}\}$ with u, v in $C^\infty(S^1)$ we define

$$a(u, v) = \int_0^{2\pi} \frac{du}{dx} \bar{v} \, dx .$$

This is well defined, a bilinear functional, which extends by continuity to $H \times H$. We define $A : H \to H^*$ by

$$(Au, v) = a(u, v) \qquad \text{for all } v \text{ in } H .$$

PROPOSITION: *Let $g \in H^*$ with $\|g\|$ sufficiently small. Then for every f in H^*, there exists $\tilde{u}$ in H such that*

$$A\tilde{u} + \|\tilde{u}\| g = f .$$

Thus, if we define $H^{1/2}(S^1)$ to be $\left\{ u \in L^2(S^1) \text{ such that } (\|u\|_{1/2})^2 = |a_0^2| + \sum_{n=-\infty}^{\infty} |na_n^2| < \infty \right\}$, for f in H^*, there exists u in $H^{1/2}(S^1)$ such that

$$\frac{du}{dx} + \inf\{\|u - k\|_{1/2} : k \text{ constant function}\}\, g = f .$$

Proof: We show that A is monotone. For u in $C^\infty(S^1)$

$$\operatorname{Re} a(u, u) = \operatorname{Re} \int_0^{2\pi} \frac{du}{dx} \cdot \bar{u} = 1/2\, [u\bar{u}]_0^{2\pi} = 0 ,$$

because u is periodic. Consequently the same holds for $\tilde{u}$ in H by continuity.

A is bijective; we prove that it is onto. For let $f \in H^*$, then $f = \sum_{n=-\infty}^{\infty} b_n e^{inx}$, $b_0 = 0$, and the element $\tilde{u} = \sum_{n=-\infty}^{\infty} (in)^{-1} b_n e^{inx} + \{\text{constants}\}$ is such that $A\tilde{u} = f$. Also, one sees that $A^*(u) = -A(\bar{u})$, so that A^* is also onto; this proves that A is one to one. Hence A^{-1} is a bounded linear operator by Banach's inverse mapping theorem. The mapping $C : H \to H^*$ defined by $C(u) = \|u\| g$ is compact.

The proof that the image under $A + C$ of each ball is closed is a consequence of the boundedness of A^{-1}.

Now let $T_t x = Ax + tCx$, x in H, $0 \le t \le 1$. We see that each T_t is quasimonotone, H 1 of Theorem 1 holds, and T_0 is an odd mapping. Moreover, if $T_t(x) = f$, we have

$$Ax = f - t\|x\| g ,$$

so that,

$$\|x\| \le \|A^{-1}\| (\|f\| + t\|x\|\,\|g\|), \quad \text{which gives, for } \|g\| < \|A^{-1}\|^{-1},$$
$$\|x\| \le \|f\|\,\|A^{-1}\|\,(1 - \|A^{-1}\|\,\|g\|)^{-1} .$$

Finally, H is a separable Hilbert space, in particular reflexive. Thus all the requirements for Theorem 2 are met and so $A + C$ is surjective, which completes the proof.

Remark. The mapping $A + C$ above is not pseudomonotone for $g \neq 0$. For, there is a sequence u_n, $\|u_n\| = 1$, and $u_n \to 0$ weakly. For such a sequence we have $\limsup\,(Au_n + Cu_n\,;\,u_n - u) = 0$. If $A+C$ were pseudomonotone this would imply that for all v in H, $\liminf\,(Au_n + Cu_n\,,\,u_n - v) \geq 0$. This gives $(g\,,\,-v) \geq 0$ for all v in H, contradicting $g \neq 0$. A similar argument can be used to show that (M) is also not satisfied.

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