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On a characterisation of k-set contractions

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Analisi funzionale. — *On a characterisation of k–set contractions* (*). Nota di JEFFREY RONALD LESLIE WEBB, presentata (**) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si dà una condizione necessaria e sufficiente affinché un operatore T sia una « (β) k–set contraction ». Quando T è lineare e la sua immagine è contenuta in uno spazio di Hilbert si ottiene un risultato più forte il quale fornisce una caratterizzazione particolarmente semplice.

In this Note we give a necessary and sufficient condition on an operator T acting between two Banach spaces in order that it be a k–set contraction with respect to the ball measure of noncompactness (see the definitions below). In the case that T is a linear mapping and the image space is Hilbert this is strengthened to give a characterisation of linear (β) k–set contractions; the proof is carried out with the aid of a new characterisation of the ball measure of noncompactness of a set in an arbitrary Banach space. The result obtained is in some sense a negative one for it shows that, in the case under study, the structure of these operators is particularly simple. The corollaries obtained regarding properties of the adjoint operator and the connection with the (α) k–set contractions have been proved earlier by this Author. We also give a sufficient condition of another kind for a linear map be a (β) k–set contraction.

Throughout X and H will stand for an arbitrary Banach space and Hilbert space respectively. We recall the notion of a measure of noncompactness. Let Ω be a bounded subset of X: the (α) measure of noncompactness of Ω is $\alpha(\Omega) = \inf\{\delta > 0$ such that there exists a finite covering by sets of diameter $\leq \delta\}$; the (β) measure is $\beta(\Omega) = \inf\{\delta > 0$ such that Ω can be covered by finitely many balls of diameter $\delta\}$. These measures are different (an example is given in [7]) though they share certain important properties, for example, if $cl(\Omega)$ and $co(\Omega)$ denote respectively the closure and the convex hull of Ω, $\beta(\Omega) = \beta(cl(\Omega)) = \beta(co(\Omega))$, with similar equalities for the other measure. The following new characterisation of the (β) measure will be of use to us: in fact, it is a strengthening of the definition.

PROPOSITION 1. *Let* Ω *be a bounded subset of* X. *Then* $\beta(\Omega) \leq 2\delta$ *if and only if for each* $\varepsilon > 0$ *there is a finite dimensional subspace* F_ε *such that, if* $B = B(0, 1)$ *denotes the closed unit ball in* X, $\Omega \subset (\delta + \varepsilon) B + F_\varepsilon$.

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Proof. The "only if" part is clear. Suppose that for $\varepsilon > 0$ there is a finite dimensional subspace F such that $\Omega \subset (\delta + \varepsilon)\,B + F$. Because Ω and B are bounded we can replace F by a bounded subset of F and this subset is relatively compact in F and hence also in X. Let this subset be denoted by G. Then, $\beta(\Omega) \leq \beta((\delta + \varepsilon)\,B) + \beta(G) = (\delta + \varepsilon)\,\beta(B) = 2\,\delta + 2\,\varepsilon$. As $\varepsilon > 0$ was arbitrary the proof is complete.

The operators we study are defined in terms of the measures of noncompactness. Let Y be a Banach space; a (not necessarily linear) map $T : D(T) \subset X \to Y$ is called a (β) k–set contraction if there is a constant $k \geq 0$ such that for all bounded subsets $\Omega \subset D(T)$ the inequality $\beta(T(\Omega)) \leq \leq k\beta(\Omega)$ holds. We have used the same symbol to define the measure of noncompactness in both X and Y even though in general they are different; we believe it will be clear at each stage which measure is meant. The (α) k–set contractions are defined analogously.

Our first result is for nonlinear operators.

THEOREM 1. *Let* $T : D(T) \subset X \to Y$ *be continuous and let* Ω *be a nonempty bounded subset of* X. *Then* $\beta(T(\Omega)) \leq 2\,k$ *if and only if for each* $\varepsilon > 0$ *there is an operator* S_ε, *whose range is finite dimensional* (*hence* S_ε *is compact*), *from* Ω *to* Y *such that* $\|T(x) - S_\varepsilon(x)\| \leq k + \varepsilon$ *for all* x *in* Ω.

Proof. The "if" statement follows directly from the compactness of S_ε and the inequality. Conversely, let $\varepsilon > 0$. Then there are points $y_1, y_2, \cdots, y_n$ such that the balls $B\left(y_j\,;\,k + \frac{1}{2}\,\varepsilon\right)$, $1 \leq j \leq n$, cover $T(\Omega)$. For x in Ω set $\mu_j(x) = \max\{0\,,\,(k + \varepsilon) - \|T(x) - y_j\|\}$. Clearly these are real valued continuous functions and for each x in Ω there is j such that $\mu_j(x) \geq \frac{1}{2}\,\varepsilon$. Therefore, $\lambda_j(x) = \frac{\mu_j(x)}{\Sigma\mu_j(x)}$ is well defined and continuous and $\lambda_j(x) = 0$ if and only if $\|T(x) - y_j\| \geq k + \varepsilon$. Set $S_\varepsilon(x) = \sum_{j=1}^{n} \lambda_j(x)\,y_j$: this is continuous and has finite dimensional range. Moreover,

$$\|T(x) - S_\varepsilon(x)\| = \|\Sigma\lambda_j(x)\,(y_j - T(x)\| \leq \sum_{j=1}^{n} \lambda_j(x)\,(k + \varepsilon) = k + \varepsilon\,,$$

for all x in Ω.

Remark. The above proof is exactly the one given to show that compact maps can be approximated uniformly by finite dimensional mappings. It applies directly to k–set contractions on the unit ball.

It would be hoped that in the linear case rather more could be said. This can be done but we require that the image space be Hilbert.

THEOREM 2. *Let* $T : X \to H$ *be a bounded linear operator. Then* $k = \inf\{l : T$ *is a* (β) l*–set contraction*$\}$ *if and only if, for each* $\varepsilon > 0$ *there is a compact linear operator* S_ε *and an operator* L_ε *with* $k \leq \|L_\varepsilon\| \leq k + \varepsilon$ *such that* $T = S_\varepsilon + L_\varepsilon$. *Moreover,* $k = \frac{1}{2}\,\beta(T(B))$, B *the unit ball in* X.

For the proof of the last part of the Theorem we state:

PROPOSITION 2. *Let* $T : X \to Y$ *be a bounded linear operator. Then* $k = \inf\{l : T$ *is a* l*-set contraction*$\}$ *if and only if* $\beta(T(B)) = 2k$.

This proposition is stated by Nussbaum [3] and essentially proved by Daneš [1]. The proof is easy and we omit it. The effective gist is that there do not exist linear mappings such that $\beta(T(\Omega)) < \beta(\Omega)$ (so called condensing maps) that are not k-set contractions for $k < 1$. This means that Theorems 11 and 12 of Petryshyn [4] do not extend results of Nussbaum [3]; likewise Theorem 5 of Webb [6] was obtained earlier by Nussbaum [3] by different methods.

Proof of Theorem 2. Let $\varepsilon > 0$; by Proposition 1, if T is a k-set contraction, $T(B_X) \subset (k+\varepsilon) B_H + F_\varepsilon$. Let P be the orthogonal projection of H onto the finite dimensional subspace F_ε. Let $S_\varepsilon = PT$; this has finite dimensional range so the proof will follow if we show that $T - PT$ has norm in the required range. Let x be in B_X, then Tx is uniquely decomposed as

$$Tx = PTx + (I - P)Tx .$$

However, for x in B_X, $Tx = b + f$, b in $(k+\varepsilon) B_H$, f in F_ε. Therefore, $(I-P)Tx = (I-P)b$ and so $\|(I-P)Tx\| \leq k + \varepsilon$ for all x in B_X, because $\|I - P\| \leq 1$. This shows that $\|(I-P)T\| \leq k + \varepsilon$. In fact, $k \leq \|(I-P)T\|$ because otherwise T would be a l-set contraction for $l < k$, an impossibility. The converse is all but obvious since a linear operator is always a k-set contraction for k less than or equal to the norm, and the sum of a compact operator with a k-set contraction is again a k-set contraction.

Remark. This result could have been obtained by Lebow and Schechter [2] if they had observed that Hilbert spaces have what they call the "compact approximation property" with constant 1; see their Theorem 3.6.

COROLLARY 1. $T : H \to X$ *is of the form* $T = S_\varepsilon + L_\varepsilon$, S_ε *compact and* $k \leq \|L_\varepsilon\| \leq k + \varepsilon$ *if and only if* T^* *is a* (β) k*-set contraction.*

As an example of Corollary 1 we give the following criterion in order that $T : H \to X$ be a (linear) (β) k-set contraction.

THEOREM 3. *Let* $T : H \to X$ *be a bounded linear operator and suppose there is a sequence* $\{f_n\}$ *in* H *with* $\|f_n\| \to k$ *such that* $\|Tx\| \leq \sup_n |(f_n, x)|$ *for all* x *in* H. *Then* T *is a* (β) k*-set contraction of the form given in Corollary* 1.

Proof. For $\varepsilon > 0$, let N be such that $\|f_n\| \leq k + \varepsilon$ for all $n > N$. Define $M_\varepsilon = \{x$ in H such that $(f_j, x) = 0$, $1 \leq j \leq N\}$. Then $T(M_\varepsilon \cap B_H) \subset$ $\subset (k+\varepsilon) B_X$. Taking polars, we leave the reader to check the following calculation:

$$(k+\varepsilon)^{-1} (B_X)^0 \subset (T^*)^{-1} (B_H \cap M_\varepsilon)^0 = (T^*)^{-1} (B_H^0 + M_\varepsilon^\perp)$$

that is,

$$\mathrm{T}^* (\mathrm{B}_{\mathrm{X}}^0) = \mathrm{T}^* (\mathrm{B}_{\mathrm{X}^*}) \subset (k + \varepsilon)\, \mathrm{B}_{\mathrm{H}} + \mathrm{M}_\varepsilon^{\perp}.$$

Appealing to Propositions 1 and 2 we see that T^* is a (β) k–set contraction; the proof is completed by an application of Corollary 1.

The case $k = 0$ was proved by Terzioğlu [5] and he showed that the converse also held in that case.

The following Corollary is obvious in view of Theorem 2. The cases (1) and (2) were established by this Author in [6] by different methods; although the results were stated in [6] only for maps $\mathrm{T} : \mathrm{H} \to \mathrm{H}$ the proofs there are also valid for the case discussed here.

COROLLARY 2. (1) *Let* $\mathrm{T} : \mathrm{X} \to \mathrm{H}$ *be a linear* (β) *k–set contraction. Then* T^* *is a* (β) *k–set contraction.*

(2) T *as in* (1) *is an* (α) *k–set contraction.*

(3) *Let* $\mathrm{T} : \mathrm{H} \to \mathrm{X}$ *be as in Corollary* 1. *Then* T *is both a* (β) *and an* (α) *k–set contraction.*

REFERENCES

[1] J. DANEŠ, *Fixed point theorems, Nemyckii and Uryson operators and continuity of non-linear mappings*, «Comment. Math. Univ. Carolinae», *11*, 481–500 (1970).

[2] A. LEBOW and M. SCHECHTER, *Semigroups of operators and measures of noncompactness*, « J. Functional Anal. », 7, 1–26 (1971).

[3] R. D. NUSSBAUM, *The fixed point index and fixed point theorems for k-set contractions*, Thesis, Chicago (1969).

[4] W. V. PETRYSHYN, *Remarks on condensing and k-set contractive mappings*, to appear in « J. Math. Anal. Appl. ».

[5] T. TERZIOĞLU, *A characterisation of compact linear mappings*, « Archiv der Mathematik », *22*, 76–78 (1971).

[6] J. R. L. WEBB, *Remarks on k-set contractions*, to appear in « Boll. Unione Mat. Ital. ».

[7] J. R. L. WEBB, *A fixed point theorem and applications to functional equations in Banach spaces*, to appear in « Boll. Unione Mat. Ital. ».