Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.6, p. 682–685.

Accademia Nazionale dei Lincei

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Analisi funzionale. — *Characteristic vectors of nonlinear operators* (*).

Nota di SIMEON REICH, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Usando alcuni recenti teoremi di punto fisso si estendono alcuni risultati riguardanti l'esistenza di vettori caratteristici di operatori non lineari definiti negli spazi normati.

Let E be a linear topological space. A closed subset $C \neq \{0\}$ of E is called a cone if

(*i*) $x, y \in C \Rightarrow ax + by \in C$ for all nonnegative a and b,

(*ii*) $x, -x \in C \Rightarrow x = 0$.

C induces a partial order in E ($x \leq y \Longleftrightarrow y - x \in C$). The elements of C are the positive elements of E.

Let S be a subset of E which has a nonempty intersection with C. A function $T : S \to E$ is said to be positive if $x \in C \cap S \Rightarrow T(x) \in C$.

In the sequel we shall restrict our attention to normed linear spaces. If $x_0 \in E$ and r is positive, $B(x_0, r)$ will denote the ball $\{x \in E \mid \|x - x_0\| \leq r\}$ while $S(x_0, r)$ will stand for its boundary, namely the set $\{x \in E \mid \|x - x_0\| = r\}$.

With any bounded $A \subset E$ one can associate a nonnegative number $m(A)$—its measure of non-compactness—as follows: $m(A) = \inf \{d > 0 \mid A$ can be covered by a finite number of its subsets with diameter not greater than $d\}$. $m(A) = 0$ if and only if A is totally bounded. If A is contained in a complete subset of E, then $m(A) = 0$ implies the compactness of the closure of A. Let conv (A) be the convex hull of A. It is not difficult to see that $m(\text{conv}(A)) = m(A)$.

$f : S \to E$ will be called condensing if for any bounded $A \subset S$ which is not totally bounded $f(A)$ is also bounded and $m(f(A)) < m(A)$. It will be called a k-set-contraction, $k \geq 0$, if for any bounded $A \subset S$, $m(f(A)) \leq km(A)$. Note that a completely continuous function is a k-set-contraction with $k = 0$. If f is a k-set-contraction for some k, we put $K(f) = \inf \{k \mid f$ is a k-set-contraction$\}$.

Although he used a slightly different notion of a measure of noncompactness, Sadovskii ([5], p. 152) has recently shown in fact that a continuous condensing self-mapping of a bounded complete convex subset of a normed

(*) AMS (MOS) Subject Classification Scheme (1970): 47H99. Key Words and Phrases: Characteristic vector, condensing function, cone, measure of non-compactness, normed linear space, positive operator.

(**) Nella seduta del 18 giugno 1971.

linear space has a fixed point (see also [1], p. 197). This result enables us to improve some theorems of Krasnosel'skii, Rothe and Schaefer which deal with characteristic vectors of (positive) operators. The reader is referred to their work for a detailed discussion of these and other results and their applications. Unfortunately, our methods are not strong enough to settle all the questions that can be raised. For example, the Birkhoff-Kellogg theorem ([3], p. 124) is not tackled.

We begin by a generalization of a result due to Morgenstern which appears on p. 849 of [6]. Recall that the norm of a normed linear space is said to be additive on a cone C if $\|x+y\| = \|x\| + \|y\|$ for all $x, y \in C$. If $T : C \to C$ and r is positive we put $p(T, r) = \inf\{\|T(x)\|/r \mid x \in C \cap S(0, r)\}$ and $P(T) = \sup\{p(T, r) \mid r > 0\}$.

THEOREM 1. *Let* E *be a normed linear space with a complete cone* C *on which the norm is additive. Let* $T : C \to C$ *be continuous and a k–set-contraction. If* $2K(T) < P(T)$, *then* T *has a positive characteristic value corresponding to a nonzero positive characteristic vector.*

Proof. Let $r > 0$ satisfy $p(T, r) > 2K(T)$. Put $V = C \cap S(0, r)$ and consider $F : V \to V$ defined by $F(x) = rT(x)/\|T(x)\|$. V is convex by the additivity of the norm. Now, if $x, y \neq 0$, then $\big\| x/\|x\| - y/\|y\| \big\| \leq$ $\leq 4\|x-y\|/(\|x\| + \|y\|)$. Consequently, F is a k–set-contraction with $k = 2K(T)/p(T, r) < 1$. By Sadovskii's theorem, F has a fixed point $z \in C$. $\|T(z)\|/r$ is the desired characteristic value.

If E happens to be an inner-product space the coefficient 2 in $2K(T) < P(T)$ can be replaced by 1 (this is so because the 4 in the norm inequality appearing in the proof can be replaced by 2).

If $1 < P(T)$ (< 2), then the condition $2K(T) < P(T)$ can be replaced by the condition $K(T) < 1$ in any normed linear space. To see this, let r satisfy $p(T, r) = 1$ and let A be a bounded subset of V. $F(A) \subset \operatorname{conv}[\{0\} \cup T(A)]$. Hence $m(F(A)) = m(T(A))$ and F is condensing. Again Sadovskii's theorem yields the required conclusion.

If for some r $M = \inf\{\|T(x)\| \mid x \in C \cap B(0, r)\}$ is positive, we may omit the assumption that the norm is additive on C and demand only that $K(T) < 1$. This is because in this case $G : C \cap B(0, r) \to C \cap B(0, r)$ defined by $G(x) = MT(x)/\|T(x)\|$ is condensing. This extends a result of Krasnosel'skii's ([2], p. 242).

We feel that Theorem 1 can be strengthened in other ways as well.

It turns out that the additivity requirement imposed on the norm can be weakened. Following Schaefer ([6], p. 850) we say that the cone C is normal if there is a positive constant $d(C)$ such that $\|x+y\| \geq d(C)\|y\|$ for all x, y which belong to C.

Let $T : C \to C$, and let $0 < r < R$. Put $q(T, r, R) = \inf\{\|T(x)\|/R \mid x \in C \cap B(0, R), \|x\| \geq r\}$, $q_1(T, R) = \sup\{q(T, r, R) \mid 0 < r < R\}$, and finally $Q(T) = \sup\{q_1(T, R) \mid R > 0\}$.

THEOREM 2 (Cf. [6], p. 850). *Let* E *be a normed linear space with a normal complete cone* C. *Let* $T : C \to C$ *be continuous and a k–set-contraction. If* $2\,K(T) < d(C)\,Q(T)$, *then* T *has a positive characteristic value corresponding to a nonzero positive characteristic vector.*

Proof. Choose $0 < r < R$ such that $q_1(T, R) > 2\,K(T)/d(C)$ and $q(T, r, R) > 2\,K(T)/d(C)$. Let $y \in C$ satisfy $\|y\| > 2\,RK(T)/[d(C)(R - r)]$ and consider the operator $S(x) = T(x) + (R - \|x\|)\,y$ defined on $W = C \cap B(0, R)$. S is a k–set-contraction with $k = K(T)$. Moreover, if $x \in W$ with $\|x\| \geq r$, then $\|T(x)\| \geq Rq(T, r, R)$ and if $x \in W$ with $\|x\| \leq r$, then $(R - \|x\|)\,\|y\| \geq (R - r)\,\|y\|$. Hence $\inf\{S(x) \mid x \in W\} \geq \geq d(C) \min\{Rq(T, r, R), (R - r)\,\|y\|\} > 2RK(T)$. It follows that $F : W \to W$ defined by $F(x) = RS(x)/\|S(x)\|$ is a k–set-contraction with $k < 1$. Therefore it has a fixed point z. This is the required characteristic vector because its norm equals R.

THEOREM 3 (Cf. [3], p. 132). *Let* E *be a normed linear space with a complete cone* C. *Then any continuous and condensing* $T : C \to C$ *with* $T(0) \neq 0$ *has a nonzero positive characteristic vector with a characteristic value greater than* 1.

Proof. Let $R > 0$ and put $W = C \cap B(0, R)$. Define $F : W \to W$ by $T(x)$ if $\|T(x)\| \leq R$ and by $RT(x)/\|T(x)\|$ if $\|T(x)\| \geq R$. By Sadovskii's theorem, F has a fixed point z. If $\|T(z)\| > R$, then $T(z) = mz$ where $m = \|T(z)\|/R > 1$ and $\|z\| = R$. If $\|T(z)\| \leq R$, then $T(z) = z$. Since there is no sequence consisting of fixed points of T which converges to 0, the result follows.

We close with an extension of a result due to Rothe ([4], p. 250). This time let E be an inner-product space. Let u be a point of $S = S(0, r)$ and let A be a closed subset of S which does not contain u. A is called convex with respect to u if the projection from u on S of every chord joining two points of A is a subset of A. It follows that the stereographic projection of A from u on the tangent plane U to S at $-u$ $(B : A \to U)$ is also convex.

THEOREM 4. *Let* $F : E_1 \to E$ *be a continuous k–set-contraction defined on a subset of* E *which contains a closed subset* A *of* $S = S(0, r)$. *Suppose that* A *is convex with respect to some point u of* S *with a distance* $d > 0$ *from* A *and that it is invariant under* G *defined by* $G(a) = rF(a)/\|F(a)\|$, $a \in A$. *If* $\|F(a)\| \geq M > 0$ *for all* $a \in A$ *and* $K(F) < Md^4/32\,r^5$, *then* F *has a positive characteristic value with a characteristic vector belonging to* A.

Proof. Let $X = B(A)$ and consider the function $H : X \to X$ defined by $H(x) = BGB^{-1}(x)$. X is convex closed and bounded. Since $B^{-1}(X)$ is contained in the convex hull of $\{u\} \cup X$, and since B is Lipschitzian with a Lipschitz constant smaller than $32r^4/d^4$ [(4], p. 248) it readily follows that H is condensing. Hence it has a fixed point $z \in X$. $B^{-1}(z) \in A$ is a fixed point of G. This latter point is a characteristic vector of F.

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