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On some classes of contractions

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Analisi funzionale. — *On some classes of contractions.* Nota di IOANA ISTRĂŢESCU presentata [(*)] dal Socio G. SANSONE.

RIASSUNTO. — Si indica una nuova classe di operatori per i quali è valida una congettura di Sr. Nagy sulle unimodulari costruzioni numeriche.

1. Let H be a Hilbert space and T be an operator which is continuous on H. In [8] Sz.-Nagy has proved that if T is completely continuous and

$$\|T^n\| < \infty \qquad n = 1, 2, \cdots$$

then there exists a contraction T′ and a self-adjoint operator Q such that

$$T' = QTQ^{-1}.$$

It is known that this result does not hold generally, but there are some cases in which it holds for other classes of operators. It is the purpose of the present note to indicate a new class for which the Sz.-Nagy conjecture is true and to give some results on unimodular numerical contractions.

2. Let us consider after Kuratowski the Kuratowski number of a bounded set in a Hilbert space H, $\alpha(A)$ defined as the infimum of all $\varepsilon > 0$ for which there exists a finite covering of A with sets of diameter less than ε.

DEFINITION 2.1. *An operator T is called α–contraction if there exists $k \in [0, 1)$ such that for all bounded sets* A *in* H *we have*

$$\alpha(TA) \leq k\alpha(A).$$

For properties of $\alpha(.)$ needed here we refer to [1].

Our main result is the following

THEOREM 2.1. *If* T *is α–contraction and*

$$\|T^n\| \leq M < \infty \qquad n = 1, 2, \cdots$$

then there exists a contraction T′ *and a self-adjoint operator* Q *such that*

$$T' = QTQ^{-1}.$$

Proof. If we denote by r_T the spectral radius of T then it is clear that $r_T \leq 1$ and thus the spectrum of T, $\sigma(T)$ lies in the unit circle. From the result in [1] we have that only a finite number of points of $\sigma(T)$ are in

$$\Gamma = \{z, |z| = 1\}.$$

(*) Nella seduta del 18 giugno 1971.

Let denote these as $\lambda_1, \lambda_2, \cdots, \lambda_n$. Thus

$$\sigma = \{\lambda_1\} \cup \cdots \cup \{\lambda_n\} \cup \sigma_0$$

and from known results [1] we can consider the idempotents $P_k = P_{\lambda_k}$ which are projections and also α–contractions. Thus the spaces $R_i = P_i H$ are of finite-dimension.

As in Sz.-Nagy paper [8] we can show for all $x \in R_i$ that we have

$$Tx = \lambda_i x .$$

The operator $T_0 = TP_0$ has the spectrum equal to σ_0 and the arguments of Sz.-Nagy show that T′ exists and also Q with the properties needed in the theorem.

COROLLARY 1. *If* T *is of the form* $T = T_1 + C$ *where* T_1 *is a* α*–contraction,* C = *compact and* T *is with uniformly bounded iterates, then* T *is similar to a contraction.*

Remark. If $\|T_1\| < 1$ we obtain Theorem 2 of [6].

COROLLARY 2. *If* T *is an operator such that for some integer* m, T^m *is* α*–contraction and with uniformly bounded iterates then* T *is similar to a contraction.*

3. We say that T is a unimodular numerical contraction if it satisfies conditions:

(*i*) T is a numerical contraction, i.e., $w_T = \sup_{\|x\|=1} |< Tx , x >| \leq 1$

(*ii*) the spectrum $\sigma(T) \subseteq \{z : |z| = 1\}$.

We denote by N(T) and R(T) the null space and range of T, $\pi_0(T)$ the set of eigenvalues of T, $\pi_{0f}(T)$ the eigenvalues of finite multiplicity, $\pi_{00}(T)$ the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.

The Weyl spectrum $\omega(T)$ of T is defined by $\omega(T) = \bigcap_K \sigma(T + K)$ where K varies over the set of all compact operators [3]. We say that Weyl's theorem holds for an operator T if

$$\omega(T) = \sigma(T) - \pi_{00}(T) .$$

First we give the following

LEMMA. *If* T *is an operator for which Weyl's theorem holds and* A *such that* $A = S^{-1} TS$ *the Weyl theorem holds for* A.

The proof follows from the fact that $\sigma(A) = \sigma(T)$ and

$$\omega(A) = \bigcap_K \sigma(A + K) = \bigcap_K \sigma(S^{-1} TS + K) = \bigcap_K \sigma(S^{-1} TS + S^{-1} SKS^{-1} S) =$$
$$= \bigcap_K \sigma(T + SKS^{-1}) = \omega(T) .$$

THEOREM 3.1. *If* T *is a unimodular numerical contraction then Weyl's theorem holds for* T.

Proof. It is known [9] that there exists an invertible operator S and an operator A, $\|A\| \leq 1$ such that $T = S^{-1} AS$. From the fact that $\sigma(A) \subseteq \{z : |z| = 1\}$, by [4] the theorem follows.

From the above lemma and a result of Sz.-Nagy [7] it follows that for any operator T with $\|T^n\| \leq M$, $n = 0, \pm 1, \pm 2, \cdots$ Weyl's theorem holds for T.

THEOREM 3.2. *Let* T *be a reduction-isoloid unimodular numerical contraction. If* T *satisfies one of the conditions*:

1) $\sigma(T)$ *is a countable set*;

2) T *is a polynomially compact operator*;

3) T *is with compact imaginary part*;

then T *is an unitary operator.*

Proof. Recall that an operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. Since T reduced by each of its eigenspaces the proof of 1) and 2) follows as in [2]. For the assertion 3), let $T = A + iB$, then $\omega(T) = \omega(A) \subseteq \{-1, 1\}$ and by theorem 3.1, Weyl's theorem holds for T. This implies that $\sigma(T)$ is a countable set and the theorem is proved.

REFERENCES

[1] A. AMBROSETTI, *Proprietà spettrali di certi operatori lineari non compatti*, « Rend. Seminario Mat. Univ. Padova », *42*, 189–200 (1969).

[2] S. K. BERBERIAN, *Some Conditions on an Operator Implying Normality*, «Math. Ann. », 184, 188–192 (1970).

[3] S. K. BERBERIAN, *The Weyl spectrum of an operator*, « Indiana Univ. Math. J. », *20*, 529–544 (1970).

[4] I. ISTRĂŢESCU, *On unimodular contractions on Banach spaces and Hilbert spaces* (to appear).

[5] I. ISTRĂŢESCU, *Unimodular numerical contractions in Hilbert spaces* (to appear).

[6] I. ISTRĂŢESCU, *Remarks concerning uniformly bounded operators on Hilbert spaces* (to appear).

[7] SZ.–NAGY B., *On uniformly bounded linear transformations in Hilbert space*, «Acta Sci. Math. (Szeged) », *11*, 152–157 (1946–48).

[8] SZ.–NAGY B., *Completely continuous operators with uniformly bounded iterates*, « Publ. Math. Hungarian Acad. Sci. », *4*, 89–93 (1959).

[9] SZ.–NAGY B. et C. FOIAŞ, *Analyse harmonique des opérateurs de l'espace de Hilbert* Masson Paris et Acad. Kiado Budapest (1967).