Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

M. Edelstein, S. Swaminathan

On the representation of mappings of normal Hausdorff spaces as restrictions of linear transformations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.6, p. 676–678.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_50_6_676_0>

Analisi funzionale. — *On the representation of mappings of normal Hausdorff spaces as restrictions of linear transformations* (*). Nota di M. Edelstein e S. Swaminathan, presentata (**) dal Socio G. Sansone.

Riassunto. — Sia X uno spazio normale di Hausdorff e f un omeomorfismo di X su di un suo sottoinsieme chiuso, e sia λ un numero reale con $0 < \lambda < 1$. Si supponga che $\bigcap_1^\infty f^n[X] = \varnothing$ $\left[\bigcap_1^\infty f^n[X]\right.$ è formato da un solo elemento$\left.\right]$; esiste allora un omeomorfismo [una applicazione continua e biunivoca] h di X in un opportuno cubo di Tichonov Q^A tale che hfh^{-1} è la restrizione ad $h[X]$ dell'applicazione $y \to \lambda y$.

Introduction

Let f be a homeomorphism of a compact Hausdorff space X into itself with the property that $\bigcap_1^\infty f^n[X]$ is a singleton. In [4] L. Janos proved that for X metrizable and for any λ, $0 < \lambda < 1$, there exists a homeomorphism h of X into a separable Hilbert space H such that hfh^{-1} is the restriction to $h[X]$ of the mapping sending each $y \in H$ to λy. This result has since been extended by the same Author [5] to compact nonmetrizable spaces by replacing H with a suitable linear topological space L. In both cases the proofs given by Janos made an essential use of a theorem of Bing [1] on the extension of metrics from a closed subset of a metrizable space to the whole space. A direct and considerably simpler proof of the main result of [4] was given in [2]. In [3] a somewhat more elaborate procedure is used to establish related results for metrizable, not necessarily compact spaces.

In the present Note we use methods similar to those of [2] and [3] to prove related results for normal Hausdorff spaces. The main result of [5] follows as a corollary.

Theorem 1. *Let* X *be a normal Hausdorff space and* f *a homeomorphism of* X *onto a closed subset of* X. *Suppose* $\bigcap_{n=1}^\infty f^n[X]$ *is a singleton* $\{x_0\}$ *and* λ *a real number* $0 < \lambda < 1$. *Then there exists a continuous one-to-one mapping* h *of* X *into* Q^A, *where* $Q = [0, 1]$ *and* A *a suitable index set, such that* hfh^{-1} *is the restriction to* $h[X]$ *of the transformation which maps* $y \in Q^A$ *into* λy.

Proof: Without restriction of generality we may assume that $X \neq \{x_0\}$. Let $\{\varphi_{a,1}\}_{a \in A}$ be the set of all continuous functions from X to Q such that

$$\varphi_{a,1}(f[X]) = 0 \quad \text{and} \quad C_a = \varphi_{a,1}^{-1}[1] \neq \varnothing .$$

(*) This research was supported by National Research Council of Canada Grants A-3999 and A-5615.

(**) Nella seduta del 18 giugno 1971.

Define

$$\overline{\varphi}_{a,1} : f[X] \cup C_a \to Q$$

by

$$\overline{\varphi}_{a,1}(x) = \begin{cases} \varphi_{a,1} f^{-1}(x) & \text{if } x \in f[X] \\ 1 & \text{if } x \in C_a . \end{cases}$$

By the Tietze extension theorem there exists a continuous function $\varphi_{a,2} : X \to Q$ which extends $\overline{\varphi}_{a,1}$. Thus $\varphi_{a,2}(f(x)) = \varphi_{a,1}(x)$ and $\varphi_{a,2}(C_a) = 1$. By induction we obtain a family of mappings $\{\varphi_{a,n}\}$ $n = 1, 2, \cdots$ of X to Q with the properties

$$\varphi_{a,n}(f(x)) = \varphi_{a,n-1}(x), \qquad n = 2, 3, \cdots, \tag{1}$$

and

$$\varphi_{a,n}(C_a) = 1 . \tag{2}$$

Set $\psi_a(x) = \frac{1-\lambda}{\lambda} \sum_{m=1}^{\infty} \lambda^m \varphi_{a,m}(x)$. It is clear that ψ is a continuous function from X to Q. From the definition of ψ it readily follows that

$$\psi_a(f^n(x)) = \lambda^n \psi_a(x) . \tag{3}$$

Let $h : X \to Q^A$ be defined by $(h(x))_a = \psi_a(x)$. Clearly h is continuous and it suffices to show that h is one-to-one. Let u and v be distinct elements of X. We may clearly assume that $x_0 \notin \{u, v\}$ so that non-negative m, n exist with $\{f^{-m}(u), f^{-n}(v)\}$ contained in $X \sim f[X]$. We may further assume that $m \leq n$. If $f^{-m}(u) = f^{-n}(v) = x$, then u and v are distinct iterates of x and, by (3), $h(v) = \lambda^{n-m} h(u) \neq h(u)$, since $m = n$ would imply $u = v$. Suppose this is not the case. Writing $u' = f^{-m}(u)$ and $v' = f^{-n}(v)$ we can find an index a in A so that $\varphi_a(u') = 1$, $\varphi_a(f[X] \cup \{v'\}) = 0$. It follows from (3) that

$$\psi_a(u) = \lambda^m \geq \lambda^n > \psi_a(v)$$

whence $h(u) \neq h(v)$. Thus h is one-to-one.

Corollary. *If in Theorem* 1, X *is compact then* $h : X \to Q^A \subset \mathbf{R}^A$ *is a homeomorphism and the main result of* [5] *follows by setting* $L = \mathbf{R}^A$. *We note that the family of pseudometrics* $\{\rho_a\}$, $a \in A$, *on* X, *obtained by setting* $\rho_a(x, y) = |p_a(h(x)) - p_a(h(y))|$ *where* $x, y \in X$ *and* $p_a(h(x)) = \psi_a(x)$ *satisfies the restatement of the main result of* [5] *mentioned above, involving pseudometrics.*

Theorem 2. *Let* X *be a normal Hausdorff space and* f *a homeomorphism of* X *onto a closed subset of* X *with* $\bigcap_{n=1}^{\infty} f^n[X] = \varnothing$. *Let* λ *be a real number with* $0 < \lambda < 1$. *Then a homeomorphism* h *of* X *into* Q^A, *where* $Q = [0, 1]$ *and* A *is a suitable index set, exists such that* hfh^{-1} *is the restriction of the mapping sending* y *to* λy.

Proof: We define A and h as in the proof of Theorem 1. Then we need only show that h is a closed mapping. Suppose, then, that F is a closed subset of X and $Y = h[F]$. Let $\{y_\alpha\}$ be a net in y converging to some $y \in h[X]$. We have to show that $y \in Y$. Suppose not and let $x = h^{-1}(y)$, $x_\alpha = h^{-1}(y_\alpha)$. Then $\{x_\alpha\}$ does not converge to x. Since $\bigcap_{n=1}^{\infty} f^n[X] = \varnothing$ there is a non-negative integer m such that $u = f^{-m}(x) \in X \sim f[X]$. Now $\{x_\alpha\}$ cannot converge to u; for otherwise $h(x_\alpha) \to h(u) \neq h(x) = y$. Let now V be an open neighborhood of u contained in $X \sim f[X]$ and such that $f^m[V] \subset X \sim F$ and $\{x_\alpha\}$ is frequently in $X \sim V$. Choose $a \in A$ such that $\psi_{a,1} : X \to Q$ is continuous, $\varphi_{a,1}[X \sim V] = 0$ and $\varphi_{a,1}(u) = 1$. Then the function

$$\psi_a = \frac{1-\lambda}{\lambda} \sum_{m=1}^{\infty} \lambda^m \varphi_{a,m}$$

vanishes for all $x_\alpha \in X \sim V$ and $\psi_\alpha(x) = \lambda^m > 0$. It follows that $\{y_\alpha\}$ does not converge to y against our assumption.

The following theorem considers the case when $\bigcap_{n=1}^{\infty} f^n[X]$ is a finite set and can be proved as in Theorem 3 of [2].

THEOREM 3. *Let f be a homeomorphism of a normal Hausdorff space* X *onto closed subset of* X *such that* $\bigcap_{n=1}^{\infty} f^n[X] = \{x_1, x_2, \cdots, x_k\}$. *Let* λ *be a real number with* $0 < \lambda < 1$ *and let* p *be the permutation of* $(1, 2, \cdots, k)$ *with the property that* $p(i) = j$ *if and only if* $f(x_i) = x_j$. *Then a continuous one-to-one mapping* h *of* X *into* $E^k \times Q^A$, *where* E^k *is the Euclidean k-dimensional space, exists such that* hfh^{-1} *is the restriction to* $h[X]$ *of the transformation which assigns to* $((x_1, x_2, \cdots, x_k), y)$ *the element* $((x_{p(1)}, x_{p(2)}, \cdots, x_{p(k)}), \lambda y)$.

Remark. We have no conclusive answer as yet to the question whether the theorems above are true for completely regular Hausdorff spaces which are not normal.

REFERENCES

[1] R. H. BING, *Extending a metric*, «Duke Math. J.», *14*, 511–519 (1947).

[2] M. EDELSTEIN, *On the representation of mappings of compact metrizable spaces as restrictions of linear transformations*, «Canad. J. Math.», *22*, 372–375 (1970).

[3] M. EDELSTEIN, *On the representation of contractive homeomorphisms as transformations in Hilbert space*, « Proc. London Math. Soc. », *22*, 462–474 (1970).

[4] L. JANOS, *Linearization of a contractive homeomorphism*, «Canad. J. Math.», *20*, 1387–1390 (1967).

[5] L. JANOS, *Topological homotheties on compact Hausdorff spaces*, « Proc. Am. Math. Soc. », *21*, 562–568 (1969).