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On the motion of a viscous incompressible fluid in a tube with permeable and deformable wall

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Analisi matematica. — *On the motion of a viscous incompressible fluid in a tube with permeable and deformable wall* (*). Nota di GIOVANNI PROUSE, presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dà un teorema di esistenza della soluzione di un problema misto per le equazioni di Navier-Stokes corrispondente al moto di un fluido viscoso incomprimibile in un tubo avente parete permeabile e deformabile.

1. – The problem we shall consider consists in the study of the motion of a viscous incompressible fluid in a tube with deformable and permeable wall; this problem is, for instance, encountered in the investigation of the flow of blood in artificial arteries.

We shall study the two-dimensional case, i.e. the plane flow of the fluid, since all the results we shall give refer to this case and cannot be extended directly to flows in three or more dimensions. In order, moreover, to avoid formal complications, we shall simplify the problem as much as possible, although the results obtained hold also for more general cases.

Let Ω_φ be the set of the x_1, x_2 plane: $\Omega_\varphi = \{0 < x_1 < l, \varphi(x_1, t) < x_2 < k\}$ with boundary Γ_φ constituted by: $\Gamma_1 = \{x_1 = 0, 0 < x_2 < k\}$, $\Gamma_2 = \{x_1 = l, 0 < x_2 < k\}$, $\Gamma'_{3,\varphi} = \{0 < x_1 < l, x_2 = \varphi(x_1, t)\}$, $\Gamma''_3 = \{0 < x_1 < l, x_2 = k\}$, where $\varphi(x_1, t)$ is a function $\in C^1(R)$ $(R = \{0 \leq x_1 \leq l, 0 \leq t \leq T\})$, with $\varphi(0, t) = \varphi(l, t) = 0$. Ω_φ is the "tube" we shall consider, which corresponds to the section of an infinite layer with the plane $x_3 = 0$. Γ_1 and Γ_2 represent the initial and final sections of the "tube", while $\Gamma'_{3,\varphi}$ and Γ''_3 are the walls, which are supposed permeable; we shall assume that Γ''_3 is rigid and that $\Gamma'_{3,\varphi}$ is deformable, its deformation depending on the pressure exerted by the fluid according to a law we shall illustrate later.

If in the tube there flows a liquid of unit density and viscosity μ subject to a force $\vec{f}(x, t)$ $(x = \{x_1, x_2\})$, the velocity $\vec{u}(x, t)$ of the liquid and its pressure $p(x, t)$ satisfy the Navier-Stokes equations

$$\text{(1.1)} \qquad \begin{aligned} &\frac{\partial u_j}{\partial t} - \mu\Delta u_j + \sum_{k=1}^{2} u_k \frac{\partial u_j}{\partial x_k} + \frac{\partial p}{\partial x_j} = f_j \qquad (j = 1, 2) \\ &\sum_{k=1}^{2} \frac{\partial u_k}{\partial x_k} = 0 . \end{aligned}$$

Our aim is to determine, under appropriate conditions, the motion of the fluid (i.e. the solution of (1.1)) and the shape of Ω_φ (i.e. the function $\varphi(x_1, t)$).

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The problem is " well posed " if we assign:

a) The shape of the tube in the absence of pressure;

b) The relation between the pressure on $\Gamma'_{3,\varphi}$ and the function $\varphi(x_1, t)$;

c) The initial conditions of the system;

d) Boundary conditions on Γ_φ, which take into account the permeability assumption on $\Gamma'_{3,\varphi} \cup \Gamma''_3 = \Gamma_{3,\varphi}$.

Let us examine separately the four conditions given above.

a) We shall assume that, in the absence of pressure, the " tube " is represented by $\Omega_0 = \{0 < x_1 < l, 0 < x_2 < k\}$.

b) Let $p(x_1, t)$ be the pressure exercised by the fluid on the wall $\Gamma'_{3,\varphi}$ and suppose that the external pressure is zero. Let, moreover, R denote the rectangle $0 \leq x_1 \leq l$, $0 \leq t \leq T$, I the interval $0 < x_1 < l$ and K the set of functions $v(x_1)$ such that

$$\text{(1.2)} \qquad K = \left\{ v(x_1) \mid v \in H_0^2(I), \left| \frac{\partial^2 v}{\partial x_1^2} \right| \leq M \text{ a.e.} \right\}.$$

The relation between $p(x_1, t)$ and $\varphi(x_1, t)$ (i.e. between the pressure and the deformation of the tube) is defined by the following two conditions:

I_1) $\varphi(t) \in L^\infty(0, T; H_0^2(I))$, $\varphi'(t) \in L^2(0, T; H_0^2(I)) \cap L^\infty(0, T; L^2(I))$, $\varphi'(t) \in K$ *a.e.*,

II_1) $\varphi(t)$ *is a solution*, $\forall v(t) \in L^2(0, T; H_0^2(I))$, *with* $v'(t) \in L^2(0, T; H^{-2}(I))$, $v(t) \in K$ *a.e.*, *of the inequality*

$$\text{(1.3)} \quad \|\varphi'(t)\|^2_{L^2(I)} + \|\varphi(t)\|^2_{H_0^2(I)} - 2(\varphi'(t), v(t))_{L^2(I)} + 2\int_0^t \{\langle v'(\eta), \varphi'(\eta)\rangle - \langle D^4\varphi(\eta), v(\eta)\rangle + \langle p(\eta), v(\eta) - \varphi'(\eta)\rangle\} d\eta \leq 0 \quad \text{a.e.},$$

where $\langle , \rangle$ denotes the duality between $H^{-2}(I)$ and $H_0^2(I)$ and we have set $D^4\varphi(t) = \left\{ \frac{\partial^4 \varphi(x_1,t)}{\partial x_1^4} ; x_1 \in I \right\}$ and assumed that $p(t) \in L^2(0, T; H^{-2}(I))$.

Conditions I_1), II_1) correspond to a weak formulation of the following classical problem. Consider the rod $0 \leq x_1 \leq l$ fixed at both ends and subject to a pressure $p(x_1, t)$ and let $\varphi(x_1, t)$ be the displacement of the point x_1 at the time t; assume, moreover, that the curvature of the rod at any point cannot vary too rapidly, i.e. that, denoting by M a constant, depending on the physical properties of the rod, we have

$$\left| \frac{\partial^3 \varphi(x_1, t)}{\partial t \, \partial x_1^2} \right| \leq M.$$

The motion of the rod is therefore governed by the equation

$$\frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial^4 \varphi}{\partial x_1^4} = p$$

in the set $R' \subseteq R$ in which $\left| \frac{\partial^3 \varphi}{\partial t \, \partial x_1^2} \right| < M$ and by the equation

$$\left| \frac{\partial^3 \varphi}{\partial t \, \partial x_1^2} \right| = M$$

in $R'' = R - R'$. Moreover, $\varphi(x_1, t)$ satisfies the initial and boundary conditions

$$\varphi(x_1, 0) = \frac{\partial \varphi(x_1, 0)}{\partial t} = \varphi(0, t) = \varphi(l, t) = \frac{\partial \varphi(0, t)}{\partial x_1} = \frac{\partial \varphi(l, t)}{\partial x_1} = 0$$

and obvious compatibility conditions on $\overline{R}' \cap \overline{R}''$.

c) The initial conditions of the system are represented by the velocity $\vec{u}(x, 0)$ of the fluid at the time $t = 0$ and by the functions $\varphi(x_1, 0)$, $\frac{\partial \varphi(x_1, 0)}{\partial t}$; as indicated in *b*), we shall assume that $\varphi(x_1, 0) = \frac{\partial \varphi(x_1, 0)}{\partial t} = 0$ i.e. that the tube is initially at rest.

d) Denoting by $\vec{\nu}_\varphi$ the exterior normal to Γ_φ, the boundary conditions are expressed by the relations

$$(1.4) \qquad \frac{1}{2} |\vec{u}(x, t)|^2 + p(x, t) = \alpha_i(x, t) \qquad (x \in \Gamma_i, i = 1, 2)$$

$$(1.5) \qquad p(x, t) = \beta(x, t) \, \vec{u}(x, t) \times \vec{\nu}_\varphi \, |\vec{u}(x, t)| \qquad (x \in \Gamma_{3,\varphi} = \Gamma'_{3,\varphi} \cup \Gamma''_3)$$

$$(1.6) \qquad |\vec{u}(x, t) \times \vec{\nu}_\varphi| = |\vec{u}(x, t)| \qquad (x \in \Gamma_\varphi).$$

Equation (1.4) assigns the value of the total energy of the fluid on the initial and final sections Γ_1 and Γ_2, while (1.6) imposes the condition that the component of the velocity tangent to Γ_φ vanishes. The permeability condition on the wall $\Gamma_{3,\varphi}$ is given by (1.5), $\beta > 0$ being a permeability coefficient, which expresses the experimental law that the velocity of the fluid through the wall, which, by (1.6), is orthogonal to the wall, is proportional to the square root of the jump of pressure: in (1.5) we have assumed that the external pressure is $= 0$.

We shall prove that the problem defined above admits, for $t > 0$ sufficiently small, a solution in an appropriate generalized sense.

Let us now define some basic functional spaces (1).

Denoting by Ω an open, bounded, connected set of the x_1, x_2 plane, satisfying the cone property, let $\mathfrak{N}(\Omega)$ be the manifold of vectors $\vec{v}(x) = \{v_1(x), v_2(x)\}$ infinitely differentiable in Ω, with null divergence and

(1) For a more detailed introduction and explanation of the functional spaces here recalled, see [1], Note I.

such that, when $x \in \partial\Omega$, $|\vec{v}(x) \times \vec{\nu}| = |\vec{v}(x)|$ ($\vec{\nu}$ exterior normal to $\partial\Omega$); let, moreover, $N^\sigma(\Omega)$ denote the closure of $\mathfrak{N}(\Omega)$ in $H^\sigma(\Omega)$ and set

$$(\vec{v}, \vec{w})_{N^0(\Omega)} = \int_\Omega \sum_{j=1}^{2} v_j(x)\, w_j(x)\, d\Omega$$

$$(\vec{v}, \vec{w})_{N^1(\Omega)} = \int_\Omega \sum_{j,k=1}^{2} \frac{\partial v_j(x)}{\partial x_k} \frac{\partial w_j(x)}{\partial x_k}\, d\Omega .$$

With such definitions, it can easily be seen that $N^0(\Omega)$ and $N^1(\Omega)$ are Hilbert spaces, with $N^1(\Omega)$ dense in $N^0(\Omega)$.

Let D(A) be the set of elements $\vec{u} \in N^1(\Omega)$ such that the linear form $\vec{v} \to (\vec{u}, \vec{v})_{N^1(\Omega)}$ is continuous in the topology of $N^0(\Omega)$; D(A) is the domain of a linear, self-adjoint, positive operator A, from D(A) to $N^0(\Omega)$, such that

$$(\vec{u}, \vec{v})_{N^1(\Omega)} = (A\vec{u}, \vec{v})_{N^0(\Omega)} = (-\Delta\vec{u}, \vec{v})_{N^0(\Omega)} \qquad \forall \vec{u} \in D(A)\ ,\ \vec{v} \in N^1(\Omega).$$

We now denote by A^σ the power of order σ ($\sigma \geq 0$) of A and by $V_\sigma(\Omega) = D(A^{\sigma/2})$ the domain of $A^{\sigma/2}$; $V_\sigma(\Omega)$ is a Hilbert space with scalar product defined by

$$(\vec{u}, \vec{v})_{V_\sigma(\Omega)} = (A^{\sigma/2}\vec{u}, A^{\sigma/2}\vec{v})_{N^0(\Omega)}$$

and is such that

$$[V_\alpha(\Omega), V_\beta(\Omega)]_\theta = V_{\alpha(1-\theta)+\beta\theta}(\Omega).$$

Identifying $V_0(\Omega)$ with its dual space $(V_0(\Omega))'$, it is possible to define the spaces $V_\sigma(\Omega)$ for $\sigma < 0$ setting

$$V_{-\sigma}(\Omega) = (V_\sigma(\Omega))'.$$

Assuming that

$$\vec{f}(t) = \{\vec{f}(x, t); x \in \Omega\} \in L^2(0, T; V_{\sigma-1}(\Omega_\varphi)), (\sigma \geq 0), \alpha_i(t) = \{\alpha_i(x, t);$$
$$x \in \Gamma_i\} \in L^2(0, T; L^2(\Gamma_i)), \beta(t) = \{\beta(x, t); x \in \Gamma_{3,\varphi}\} \in L^\infty(0, T; L^\infty(\Gamma_{3,\varphi})\},$$
$$\varphi(x_1, t) \in C^1(R)$$

and setting

$$b_\varphi(\vec{u}(t), \vec{v}(t), \vec{w}(t)) = \int_{\Omega_\varphi} \sum_{j,k=1}^{2} u_j(x, t) \frac{\partial v_k(x, t)}{\partial x_j} w_k(x, t)\, d\Omega_\varphi ,$$

we shall, following the definition given in [1], *say that* $\vec{u}(t) = \{\vec{u}(x, t); x \in \Omega_\varphi\}$

is a solution in $[0, T]$ *of system* (1.1) *satisfying the boundary conditions* (1.4), (1.5), (1.6) *if*:

$\text{I}_2)$ $\vec{u}(t) \in L^2(0,T;V_{\sigma+1}(\Omega_\varphi)) \cap L^\infty(0,T;V_\sigma(\Omega_\varphi)) \cap H^1(0,T;V_{\sigma-1}(\Omega_\varphi))$;

$\text{II}_2)$ $\vec{u}(t)$ *satisfies* $\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_\varphi))$, *the equation*

$$\int_0^T \{\langle \vec{u}'(t) + \mu A\vec{u}(t) - \vec{f}(t), \vec{h}(t)\rangle_\varphi + b_\varphi(\vec{u}(t), \vec{u}(t), \vec{h}(t))\} \, dt = \tag{1.7}$$
$$= -\int_0^T \Big\{ \sum_{i=1}^2 \int_{\Gamma_i} (\alpha_i(x,t) - \frac{1}{2} u_1^2(x,t)) \vec{h}(x,t) \times \vec{\nu}_\varphi \, d\Gamma_i +$$
$$+ \int_{\Gamma_{3,\varphi}} \beta(x,t) \vec{u}(x,t) \times \vec{\nu}_\varphi |\vec{u}(x,t)| \vec{h}(x,t) \times \vec{\nu}_\varphi \, d\Gamma_{3,\varphi} \Big\} \, dt$$

where $\langle , \rangle_\varphi$ denotes the duality, $\forall t \in [0,T]$, between $V_{\sigma-1}(\Omega_\varphi)$ and $V_{1-\sigma}(\Omega_\varphi)$. As explained in greater detail in [1] relation (1.7) is obtained multiplying the first of (1.1) by h_j, adding, integrating over Ω_φ and $[0, T]$ and bearing in mind the second of (1.1) and (1.4), (1.5), (1.6).

It is obvious that the flow in the deformable tube is determined by the solution of the system, in the unknown functions $\vec{u}(t), \varphi(t)$,

$$\int_0^T \{\langle \vec{u}'(t) + \mu A\vec{u}(t) - \vec{f}(t), \vec{h}(t)\rangle_\varphi + b_\varphi(\vec{u}(t), \vec{u}(t), \vec{h}(t))\} \, dt = \tag{1.8}$$
$$= -\int_0^T \Big\{ \sum_{i=1}^2 \int_{\Gamma_i} (\alpha_i(x,t) - \frac{1}{2} u_1^2(x,t)) \vec{h}(x,t) \times \vec{\nu}_\varphi \, d\Gamma_i +$$
$$+ \int_{\Gamma_{3,\varphi}} \beta(x,t) \vec{u}(x,t) \times \vec{\nu}_\varphi |\vec{u}(x,t)| \vec{h}(x,t) \times \vec{\nu}_\varphi \, d\Gamma_{3,\varphi} \Big\} \, dt$$

$$\|\varphi'(t)\|^2_{L^2(I)} + \|\varphi(t)\|^2_{H_0^2(I)} - 2(\varphi'(t), v(t))_{L^2(I)} + 2\int_0^t \{\langle v'(\eta), \varphi'(\eta)\rangle - \langle D^4\varphi(\eta), v(\eta)\rangle +$$
$$+ \int_0^l \beta(x_1, \varphi(x_1,\eta), \eta) \, u(x_1, \varphi(x_1,\eta), \eta) \times \vec{\nu}_\varphi |\vec{u}(x_1, \varphi(x_1,\eta), \eta)| \cdot$$
$$\cdot \left(v(x_1,\eta) - \frac{\partial\varphi(x_1,\eta)}{\partial\eta}\right) dx_1 \Big\} \, d\eta \leq 0 \qquad \text{a.e.}$$

$\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_\varphi))$, $v(t) \in L^2(0, T; H_0^2(I))$, with $v(t) \in K$ a.e., $v'(t) \in L^2(0, T; H^{-2}(I))$, where $\vec{u}(t)$, $\varphi(t)$ satisfy respectively conditions $\text{I}_2)$ and $\text{I}_1)$ and $\vec{u}(t)$ is such that

$$\vec{u}(0) = \vec{u}_0 . \tag{1.9}$$

2. – Let us now give two auxiliary theorems which will be utilized in § 3 for proving the final existence theorem.

THEOREM 1. – *If* $\vec{f}(t) \in L^2(0, T; V_{\sigma-1}(\Omega_\varphi))$, $\alpha_i(t) \in L^2(0, T; L^2(\Gamma_i))$ $(i = 1, 2)$, $\beta(t) \in L^\infty(0, T; L^\infty(\Gamma_{3,\varphi}))$, $\varphi(x_1, t) \in C^1(R)$, $\vec{u}_0 \in V_\sigma(\Omega_0)$ *and if* $\frac{1}{4} < \sigma < \frac{1}{2}$ *and* T *is sufficiently small, there exists one, and only one function* $\vec{u}(t)$ *satisfying* I_2), II_2) *and the initial condition* (1.9).

The proof of this theorem (which actually holds under more general assumptions on Ω_φ than those made above) is given in [1].

THEOREM 2. – *If* $p(t) \in L^2(0, T; H^{-2}(I))$, *there exists a function* $\varphi(t)$ *satisfying* I_1), II_1).

We recall (Lions [2], Ch. 3, § 5.2) that, since the set K defined by (1.7) is a closed convex set of $H_0^2(I)$, we can associate to K an operator β, monotone, bounded and continuous from $H_0^2(I)$ to $H^{-2}(I)$ such that

$$K = \{v(x_1) \mid v \in H_0^2(I), \beta(v) = 0\} \tag{2.1}$$

Observing that D^4 is a " duality operator " from $H_0^2(I)$ to $H^{-2}(I)$ and denoting by P_K the operator, from $H_0^2(I)$ to K, " projection on K ", we have moreover, in our case,

$$\beta(v) = D^4(v - P_K v) \tag{2.2}$$

and, $\forall w \in H_0^2(I)$,

$$\langle D^4 w, w \rangle = \| D^4 w \|_{H^{-2}(I)} \| w \|_{H_0^2(I)} = \| w \|^2_{H_0^2(I)}, \tag{2.3}$$

$$\| P_K w \|_{H_0^2(I)} \leq M_0 . \tag{2.4}$$

From the definitions given it also follows that, $\forall w \in H_0^2(I)$, $z \in K$

$$\langle D^4(w - P_K w), z - P_K w \rangle \leq 0 . \tag{2.5}$$

Let $\{g_j\}$ be a basis in $H_0^2(I)$; setting

$$\varphi_n(t) = \sum_{j=1}^{n} \alpha_{jn}(t) g_j , \tag{2.6}$$

we consider the system

$$(\varphi_n''(t), g_j)_{L^2(I)} + (\varphi_n(t), g_j)_{H_0^2(I)} + n \langle \beta(\varphi_n'(t)), g_j \rangle = \langle p(t), g_j \rangle \qquad (j = 1, \cdots, n) \tag{2.7}$$

which, as can easily be verified, admits, $\forall n$, a solution, defined in a neighbourhood of $t = 0$, $\varphi_n(t)$, satisfying the initial conditions $\varphi_n(0) = \varphi_n'(0) = 0$. We now obtain some a priori estimates which ensure the existence of the solution on the whole interval $[0, T]$.

Multiplying (2.7) by $\alpha'_{jn}(t)$ and adding we have

$$(2.8) \qquad \frac{1}{2}\frac{d}{dt}\left(\|\varphi'_n(t)\|^2_{L^2(I)} + \|\varphi_n(t)\|^2_{H^2_0(I)}\right) + n\langle\beta(\varphi'_n(t)), \varphi'_n(t)\rangle = \langle p(t), \varphi'_n(t)\rangle,$$

from which follows, integrating between 0 and $t \in [0, T]$, and bearing in mind the initial conditions,

$$(2.9) \qquad \|\varphi'_n(t)\|^2_{L^2(I)} + \|\varphi_n(t)\|^2_{H^2_0(I)} + 2n\int_0^t \langle\beta(\varphi'_n(\eta)), \varphi'_n(\eta)\rangle\, d\eta \leq 2\int_0^t \|p(\eta)\|_{H^{-2}(I)} \|\varphi'_n(\eta)\|_{H^2_0(I)}\, d\eta.$$

We have, on the other hand, by (2.2), (2.3), (2.4), (2.5).

$$(2.10) \qquad \langle\beta(w), w\rangle = \langle D^4(w - P_K w), w - P_K w\rangle + \langle D^4(w - P_K w), P_K w\rangle \geq \|w - P_K w\|^2_{H^2_0(I)} \geq \|w\|^2_{H^2_0(I)} + \|P_K w\|^2_{H^2_0(I)} - 2\|w\|_{H^2_0(I)}\|P_K w\|_{H^2_0(I)} \geq \|w\|^2_{H^2_0(I)} - 2M_0\|w\|_{H^2_0(I)}.$$

Hence, introducing (2.10) in (2.9),

$$(2.11) \qquad \|\varphi'_n(t)\|^2_{L^2(I)} + \|\varphi_n(t)\|^2_{H^2_0(I)} + 2n\int_0^t \|\varphi'_n(\eta)\|_{H^2_0(I)}\left(\|\varphi'_n(\eta)\|_{H^2_0(I)} - 2M_0\right) d\eta \leq 2\int_0^t \|p(\eta)\|_{H^{-2}(I)}\|\varphi'_n(\eta)\|_{H^2_0(I)}\, d\eta.$$

From (2.11) it follows that

$$(2.12) \qquad \sup_{0\leq t\leq T}\|\varphi'_n(t)\|_{L^2(I)} \leq M_1 \quad , \quad \sup_{0\leq t\leq T}\|\varphi_n(t)\|_{H^2_0(I)} \leq M_2, \qquad \int_0^T \|\varphi'_n(t)\|^2_{H^2_0(I)}\, dt \leq M_3,$$

with M_j independent of n.

It is therefore possible to extract from $\{\varphi_n\}$ a subsequence (again denoted by $\{\varphi_n\}$) such that

$$(2.13) \qquad \lim_{n\to\infty} \varphi_n(t) \underset{L^\infty(0,T;H^2_0(I))}{=} \varphi(t) \quad , \quad \lim_{n\to\infty} \varphi'_n(t) \underset{L^\infty(0,T;L^2(I))}{=} \varphi'(t)$$

in the weak-star topologies and

(2.14) $$\lim_{n\to\infty} \varphi_n'(t) \underset{L^2(0,T;H_0^2(I))}{=} \varphi'(t),$$

in the weak topology.

From (2.9) it also follows, bearing in mind (2.2), (2.3), (2.5), (2.12), that

(2.15) $$\frac{M_4}{n} \geq \int_0^T \langle \beta(\varphi_n'(t)), \varphi_n'(t)\rangle \, dt = \int_0^T \langle D^4(\varphi_n'(t) - P_K \varphi_n'(t)), \varphi_n'(t)\rangle \, dt =$$

$$= \int_0^T \langle D^4(\varphi_n'(t) - P_K \varphi_n'(t)), \varphi_n'(t) - P_K \varphi_n'(t)\rangle \, dt +$$

$$+ \int_0^T \langle D^4(\varphi_n'(t) - P_K \varphi_n'(t)), P_K \varphi_n'(t)\rangle \, dt \geq$$

$$\geq \int_0^T \| \varphi_n'(t) - P_K \varphi_n'(t)\|^2_{H_0^2(I)} \, dt$$

and, consequently,

(2.16) $$\lim_{n\to\infty} \| \varphi_n'(t) - P_k \varphi_n'(t)\|_{L^2(0,T;H_0^2(I))} = 0 .$$

that is

(2.17) $$\lim_{n\to\infty} \varphi_n'(t) - P_k \varphi_n'(t) \underset{L^2(0,T;H_0^2(I))}{=} 0 .$$

Hence

(2.18) $$\lim_{n\to\infty} \beta(\varphi_n'(t)) = \lim_{n\to\infty} D^4(\varphi_n'(t) - P_k \varphi_n'(t)) \underset{L^2(0,T;H^{-2}(I))}{=} 0 .$$

On the other hand, by (2.14),

(2.19) $$\lim_{n\to\infty} \beta(\varphi_n'(t)) \underset{L^2(0,T;H^{-2}(I))}{=} \beta(\varphi'(t))$$

in the weak topology and we have therefore

(2.20) $$\beta(\varphi'(t)) = 0 .$$

Hence, bearing in mind (2.1),

(2.21) $$\varphi'(t) \in K \qquad \text{a.e.} \quad on\ [0, T] .$$

Let $v(t)$ be any function $\in C^1(0, T; H_0^2(I))$, with $v \in K$ a.e. and set

(2.22) $$v(t) = \sum_{j=1}^{\infty} \zeta_j(t) g_j \quad , \quad v_n(t) = \sum_{j=1}^{n} \zeta_j(t) g_j .$$

Multiplying (2.7) by $\zeta_j(t) + \alpha_j'(t)$, adding and integrating, we obtain, since $\beta(v_n) = 0$,

$$\int_0^t \langle \varphi_n''(\eta) + D^4 \varphi_n(\eta) - p(\eta), v_n(\eta) - \varphi_n'(\eta)\rangle \, d\eta = \tag{2.23}$$

$$= n \int_0^t - \langle \beta(\varphi_n'(\eta)), v_n(\eta) - \varphi_n'(\eta)\rangle \, d\eta =$$

$$= n \int_0^t \langle \beta(v_n(\eta)) - \beta(\varphi_n'(\eta)), v_n(\eta) - \varphi_n'(\eta)\rangle \, d\eta .$$

Hence, β being monotone,

$$\| \varphi_n'(t)\|^2_{L^2(I)} + \| \varphi_n(t)\|^2_{H_0^2(I)} - 2(\varphi_n'(t), v_n(t))_{L^2(I)} + \tag{2.24}$$

$$+ 2\int_0^t \{\langle v_n'(\eta), \varphi_n'(\eta)\rangle - \langle D^4 \varphi_n(\eta), v_n(\eta)\rangle + \langle p(\eta), v_n(\eta) - \varphi_n'(\eta)\rangle\} \, d\eta \leq 0 .$$

Let $\vartheta(t)$ be an arbitrary function $\in L^1(0, T)$, with $\vartheta \geq 0$; it follows from (2.24) that

$$\int_0^T (\| \varphi_n'(t)\|^2_{L^2(I)} + \| \varphi_n(t)\|^2_{H_0^2(I)}) \vartheta(t) \, dt - 2\int_0^T (\varphi_n'(t), v_n(t))_{L^2(I)} \vartheta(t) \, dt +$$

$$+ 2\int_0^T \vartheta(t) \int_0^t \{\langle v_n'(\eta), \varphi_n'(\eta)\rangle - \langle D^4 \varphi_n(\eta), v_n(\eta)\rangle +$$

$$+ \langle p(\eta), v_n(\eta) - \varphi_n'(\eta)\rangle\} \, d\eta \, dt \leq 0 .$$

Letting $n \to \infty$, we have then, by (2.13),

$$\int_0^T (\| \varphi'(t)\|^2_{L^2(I)} + \| \varphi(t)\|^2_{H_0^2(I)}) \vartheta(t) \, dt - 2\int_0^T (\varphi(t), v(t))_{L^2(I)} \vartheta(t) \, dt + \tag{2.25}$$

$$+ 2\int_0^T \vartheta(t) \int_0^t \{\langle v'(\eta), \varphi'(\eta)\rangle - \langle D^4 \varphi(\eta), v(\eta)\rangle +$$

$$+ \langle p(\eta), v(\eta) - \varphi'(\eta)\rangle\} \, d\eta \, dt \leq 0 .$$

Since the space of the functions $v(t)$ given by (2.22) is dense in that of the test functions of (1.7) and $\vartheta \geq 0$ is arbitrary, from (2.25) it follows that $\varphi(t)$ satisfies II_1); by (2.13), (2.14) $\varphi(t)$ satisfies also condition I_1) and the theorem is therefore proved.

3. – Let us, finally, prove the following theorem.

THEOREM 3. – *If* $\|\vec{f}(t)\|_{L^2(0,T;V_{\sigma-1}(\Omega_\varphi))} \leq M_1$, $\|\beta(t)\|_{L^\infty(0,T;L^\infty(\Gamma_{3,\varphi}))} \leq M_2$ $\forall\varphi(t)$ *satisfying* $I_1)$, $\alpha_i(t) \in L^2(0,T;L^2(\Gamma_i))$, $\vec{u}_0 \in V_\sigma(\Omega_0)$, *then system* (1.8) *admits, for* T *sufficiently small, a solution satisfying the initial condition* (1.9), *provided* $\frac{1}{4} < \sigma < \frac{1}{2}$.

We introduce the transformation $\varphi = S(\psi)$, from $H^1_{0-}(0,T;H^2_0(I))$ to itself $(H^1_{0-}(0,T;H^2_0(I)) = \{v \mid v(t) \in H^1(0,T;H^2_0(I)), v(0) = 0\})$, with $S = \prod_{i=1}^{3} S_i$ defined in the following way.

a) $\tilde{\psi} = S_1(\psi)$, with $\psi(t) \in H^1_{0-}(0,T;H^2_0(I))$, $\tilde{\psi}(t) \in H^1_{0-}(0,T;H^2_0(I))$, $\tilde{\psi}'(t)$ projection on K a.e. (K defined by (1.2)) of $\psi'(t)$.

b) $\vec{u} = S_2(\tilde{\psi})$, with $\vec{u}(t)$ satisfying conditions $I_2)$, $II_2)$ (in which $\varphi(t)$ has been substituted by $\tilde{\psi}(t)$) and the initial condition (1.9). The existence, $\forall\tilde{\psi}(t)$, of such $\vec{u}(t)$ is guaranteed by theorem 1. Observe that by the assumptions made, we have

$$\beta(x_1, \tilde{\psi}(x_1,t), t)\, \vec{u}(x_1, \tilde{\psi}(x_1,t), t) \times \vec{\nu}_{\tilde{\psi}} \,|\vec{u}(x_1, \tilde{\psi}(x_1,t), t)| \in L^2(R). \tag{3.1}$$

c) $\varphi = S_3(\vec{u})$, with $\varphi(t)$ satisfying $I_1)$ and the inequality

$$\begin{aligned} &\|\varphi'(t)\|^2_{L^2(I)} + \|\varphi(t)\|^2_{H^2_0(I)} - 2(\varphi'(t), v(t))_{L^2(I)} + \\ &+ 2\int_0^t \Big\{\langle v'(\eta), \varphi'(\eta)\rangle - \langle D^4\varphi(\eta), v(\eta)\rangle + \\ &+ \int_0^l \beta(x_1, \tilde{\psi}(x_1,\eta), \eta)\, \vec{u}(x_1, \tilde{\psi}(x_1,\eta), \eta) \times \vec{\nu}_{\tilde{\psi}} \,|\vec{u}(x_1, \tilde{\psi}(x_1,\eta), \eta)| \cdot \\ &\cdot \left(v(x_1,\eta) - \frac{\partial\varphi(x_1,\eta)}{\partial\eta}\right) dx_1 \Big\}\, d\eta \leq 0 \qquad \text{a.e.} \end{aligned} \tag{3.2}$$

$\forall v(t) \in L^2(0,T;H^2_0(I))$, with $v(t) \in K$ a.e., $v'(t) \in L^2(0,T;H^{-2}(I))$. By (3.1) and theorem 2, such a function exists.

It is now evident that *the problem of finding a solution of system* (1.8) *is equivalent to that of proving the existence of a fixed point* φ^* *for the transformation* $\varphi = S(\psi)$.

Let us prove, at first, that S is weakly continuous, i.e. that, if $\{\psi_n\}$ is a sequence such that

$$\lim_{n\to\infty} \psi_n(t) \underset{H^1(0,T;H^2_0(I))}{=} \psi(t) \tag{3.3}$$

in the weak topology, then it is possible to select from $\{\psi_n\}$ a subsequence (again denoted by $\{\psi_n\}$) such that

$$\lim_{n\to\infty} S(\psi_n(t)) = \lim_{n\to\infty} \varphi_n(t) \underset{H^1(0,T;H_0^2(I))}{=} \varphi(t) \tag{3.4}$$

in the weak topology, with

$$\varphi = S(\psi). \tag{3.5}$$

We observe, at first, that, as may be verified directly, S_1 is weakly continuous. Observe moreover that, setting $\vec{u}_n = S_2(\tilde{\psi}_n) = S_1 S_2(\psi_n)$, it follows from the proof of Theorem 1 and the assumptions made that

$$\|\vec{u}_n(t)\|_{L^2(0,T;V_{\sigma+1}(\Omega_{\tilde{\psi}_n}))\cap L^\infty(0,T;V_\sigma(\Omega_{\tilde{\psi}_n}))\cap H^1(0,T;V_{\sigma-1}(\Omega_{\tilde{\psi}_n}))} \leq M_1 \tag{3.6}$$

with M_1 independent of n (in the sequel, we shall always denote by M_j quantities independent of n).

Hence

$$\|\vec{u}_n(t)\|_{H^\theta(0,T;V_{\sigma+1-2\theta}(\Omega_{\tilde{\psi}_n}))} \leq M_2$$

and, choosing $\theta = \frac{1}{4} + \varepsilon\ (\varepsilon > 0)$,

$$\|\vec{u}_n(t)\|_{H^{(1/4)+\varepsilon}(0,T;V_{\sigma+(1/2)-2\varepsilon}(\Omega_{\tilde{\psi}_n}))} \leq M_2. \tag{3.7}$$

From (3.7) it follows that

$$\|\vec{u}_n(x_1, \tilde{\psi}_n(x_1, t), t)\|_{L^4(R)} \leq M_3. \tag{3.8}$$

Setting $x_1 = \xi_1$, $x_2 = \xi_2 \dfrac{k - \tilde{\psi}_n(\xi_1, t)}{k} + \tilde{\psi}_n(\xi_1, t)$, $\vec{\tilde{u}}_n(\xi_1, \xi_2, t) = \vec{u}_n\left(\xi_1, \xi_2 \dfrac{k - \tilde{\psi}_n(\xi_1, t)}{k} + \tilde{\psi}_n(\xi_1, t), t\right)$ (so that the functions $\vec{\tilde{u}}_n$ are defined for $0 < \xi_1 < l$, $0 < \xi_2 < k$, $0 \leq t \leq T$, we obtain from (3.6), (3.7) (since the functions $\tilde{\psi}_n$, $\dfrac{\partial\tilde{\psi}_n}{\partial t}$, $\dfrac{\partial\tilde{\psi}_n}{\partial x_1}$, $\dfrac{\partial^2\tilde{\psi}_n}{\partial x_1^2}$, $\dfrac{\partial^3\tilde{\psi}_n}{\partial t\,\partial x_1^2}$ are, by the assumptions made, uniformly bounded),

$$\|\vec{\tilde{u}}_n(t)\|_{L^2(0,T;V_{\sigma+1}(\Omega_0))\cap L^\infty(0,T;V_\sigma(\Omega_0))\cap H^1(0,T;V_{\sigma-1}(\Omega_0))} \leq M_4 \tag{3.9}$$

$$\|\vec{\tilde{u}}_n(t)\|_{H^{(1/4)+\varepsilon}(0,T;V_{\sigma+(1/2)-2\varepsilon}(\Omega_0))} \leq M_5.$$

It is therefore possible to select from $\{\vec{\tilde{u}}_n\}$ a subsequence (again denoted by $\{\vec{\tilde{u}}_n\}$) such that

$$(3.10) \qquad \lim_{n\to\infty} \vec{\tilde{u}}_n(t) \underset{L^2(0,T;V_{\sigma+1}(\Omega_0))\cap H^1(0,T;V_{\sigma-1}(\Omega_0))}{=} \vec{\tilde{u}}(t),$$

$$\lim_{n\to\infty} \vec{\tilde{u}}_n(t) \underset{L^\infty(0,T;V_\sigma(\Omega_0))}{=} \vec{\tilde{u}}(t)$$

in the weak and weak-star topologies respectively.

Since $\sigma > \frac{1}{4}$, the embedding

$$H^{(1/4)+\varepsilon}(0,T;V_{\sigma+(1/2)-2\varepsilon}(\Omega_0)) \subset H^{(1/4)+\varepsilon}(0,T;V_{3/4}(\Omega_0)) \subset L^4(0,T;V_{3/4}(\Omega_0))$$

is completely continuous and we can therefore assume that

$$(3.11) \qquad \lim_{n\to\infty} \vec{\tilde{u}}_n(\xi_1,0,t) \underset{L^4(R)}{=} \vec{\tilde{u}}(\xi_1,0,t)$$

in the strong topology.

On the other hand, bearing in mind that $\tilde{\psi}_n'(t) \in K$,

$$(3.12) \qquad \lim_{n\to\infty} \frac{\partial\tilde{\psi}_n}{\partial x_1} \underset{C^0(R)}{=} \frac{\partial\tilde{\psi}}{\partial x_1}$$

in the strong topology and

$$(3.13) \qquad \lim_{n\to\infty} \frac{\partial\tilde{\psi}_n}{\partial t} \underset{L^\infty(R)}{=} \frac{\partial\tilde{\psi}}{\partial t} \quad , \quad \lim_{n\to\infty} \frac{\partial^2\tilde{\psi}_n}{\partial x_1^2} \underset{L^\infty(R)}{=} \frac{\partial^2\tilde{\psi}}{\partial x_1^2}$$

in the weak-star topology, with $\tilde{\psi} = S_1(\psi)$ (by the weak continuity of S_1).

Setting $\vec{u}(x_1,x_2,t) = \vec{\tilde{u}}\left(x_1, k\frac{x_2-\tilde{\psi}(x_1,t)}{k-\tilde{\psi}(x_1,t)}, t\right)$, it can then be directly verified, by (3.10), (3.11), (3.12), (3.13), that $\vec{u}(t)$ satisfies equation (1.7), with $\varphi = \tilde{\psi} = S_1(\psi)$, ψ defined by (3.3).

By (3.8) and the proof of theorem 2, the sequence $\{\varphi_n(t)\}$ of solutions of the inequalities

$$(3.14) \qquad \|\varphi_n'(t)\|^2_{L^2(I)} + \|\varphi_n(t)\|^2_{H_0^2(I)} - 2(\varphi_n'(t), v(t))_{L^2(I)} +$$

$$+ 2\int_0^t \Big\{ \langle v'(\eta), \varphi_n'(\eta)\rangle - \langle D^4\varphi(\eta), v(\eta)\rangle +$$

$$+ \int_0^l \beta(x_1,\tilde{\psi}_n(x_1,\eta),\eta)\, \vec{u}_n(x_1,\tilde{\psi}_n(x_1,\eta),\eta) \times \vec{\nu}_{\tilde{\psi}_n} |\vec{u}_n(x_1,\tilde{\psi}_n(x_1,\eta),\eta)| \cdot$$

$$\cdot \left(v(x_1,\eta) - \frac{\partial\varphi_n(x_1,\eta)}{\partial\eta}\right) dx_1 \Big\}\, d\eta \le 0$$

is such that $\varphi'_n(t) \in K$ a.e. and

$$(3.15) \qquad \|\varphi_n(t)\|_{L^\infty(0,T;H_0^2(I))} \leq M_6 \quad , \quad \|\varphi'_n(t)\|_{L^2(0,T;H_0^2(I)) \cap L^\infty(0,T;L^2(I))} \leq M_7 .$$

Hence, analogously to (2.13), (2.14),

$$(3.16) \qquad \lim_{n\to\infty} \varphi_n(t) \underset{L^\infty(0,T;H_0^2(I))}{=} \varphi(t) \quad , \quad \lim_{n\to\infty} \varphi'_n(t) \underset{L^\infty(0,T;L^2(I))}{=} \varphi'(t)$$

in the weak-star topologies and

$$(3.17) \qquad \lim_{n\to\infty} \varphi'_n(t) \underset{L^2(0,T;H_0^2(I))}{=} \varphi'(t)$$

in the weak topology. Relation (3.4) is therefore proved.

If now in (3.14) we let $n \to \infty$, we obtain (3.2); this can be proved bearing in mind (3.8), (3.12), (3.16), by the same procedure used to deduce (1.3) from (2.24). Since $\varphi(t)$ obviously satisfies also condition I_1), we conclude that $\varphi(t)$ is a solution corresponding to $\vec{u}(t)$, i.e. $\varphi = S_3(u) = S_3 S_2(\tilde{\psi}) = S_3 S_2 S_1(\psi) = S(\psi)$. The weak continuity of S is therefore proved.

Finally, we observe that, since $\varphi(t)$ satisfies condition I_1), when $\|\psi(t)\|_{H^1(0,T;H_0^2(I))}$ is sufficiently large we have

$$(3.18) \qquad \|\varphi(t)\|_{H^1(0,T;H_0^2(I))} < \|\psi(t)\|_{H^1(0,T;H_0^2(I))} .$$

By Tychonoff's fixed point theorem, there exists then φ^* such that $\varphi^* = S(\varphi^*)$. The theorem is therefore proved.

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