Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.6, p. 650–653.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_50_6_650_0>

Analisi matematica. — *A Goursat problem for a high order Mangeron equation.* Nota di MEHMET NAMIK OĞUZTÖRELI (*) (**) e SUNDARAM EASWARAN (*), presentata (***) dal Socio M. PICONE.

RIASSUNTO. — In questo articolo viene studiato un problema caratteristico di Goursat per un'equazione lineare di ordine superiore del tipo di Mangeron con coefficienti costanti.

1. Higher order Mangeron equations play an important role in certain multidimensional approximation and mapping processes considered by G. Birkhoff, W. Gordon and M. Picone: [1]–[4]. In [7] and [8], D. Mangeron and the first of the authors investigated Goursat problems for certain Mangeron polyvibrating equations of order four and six with analytic data. In the present paper we consider a characteristic Goursat problem for a higher order linear Mangeron equation with constant coefficients.

Consider the n–th order polyvibrating equation of Mangeron

$$\partial^n u + a_1 \partial^{n-1} u + \cdots + a_{n-1} \partial u + a_n u = 0 , \tag{1.1}$$

where $a_1 , a_2 , \cdots , a_n$ are constants, ∂ is the "total derivative" operator in the sense of M. Picone [5]–[6]:

$$\partial = \frac{\partial^2}{\partial x \, \partial y} , \tag{1.2}$$

and $u = u(x , y)$ is a real valued function of the real variables x and y. We wish to establish the solution of Eq (1.1) satisfying the conditions

$$\begin{aligned} & u \big|_{x=\xi} = u \big|_{y=\eta} = 1 \\ & \partial^k u \big|_{x=\xi} = \partial^k u \big|_{y=\eta} = 0 \qquad (k = 1 , 2 , \cdots , n-1) , \end{aligned} \tag{1.3}$$

where ξ and η are two arbitrary numbers. The Goursat problem (1.1)–(1.3) occurs in the construction of the Riemann function associated with certain polyvibrating equations.

Clearly, any solution of the above problem, if there exists, is analytic in the neighborhood of the point (ξ , η), and therefore it is unique. In the following, we shall establish the solution and illustrate our approach by an example.

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(**) This work was partly supported by the National Research Council of Canada under the Grant NRC–A–4345 through the University of Alberta.
(***) Nella seduta del 18 giugno 1971.

2. We now seek a solution to problem (1.1)–(1.3) in the form $u = u(t)$, where

$$t = -(x - \xi)(y - \eta) . \tag{2.1}$$

By direct calculations and by induction we can show that

$$
\begin{aligned}
\partial u &= -\left[\frac{du}{dt} + t\frac{d^2u}{dt^2}\right] \equiv -\frac{d}{dt}\left[t\frac{du}{dt}\right],\\
\partial^2 u &= 2\frac{d^2u}{dt^2} + 4t\frac{d^3u}{dt^3} + t^2\frac{d^4u}{dt^4} \equiv -\frac{d^2}{dt^2}\left[t^2\frac{d^2u}{dt^2}\right],\\
\partial^3 u &= -\left[6\frac{d^3u}{dt^3} + 18t\frac{d^4u}{dt^4} + 9t^2\frac{d^5u}{dt^5} + t^3\frac{d^6u}{dt^6}\right] \equiv \frac{d^3}{dt^3}\left[t^3\frac{d^3u}{dt^3}\right],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\partial^k u &= (-1)^k\sum_{j=0}^{k} b_{k,j}\, t^j\frac{d^{k+j}u}{dt^{k+j}} \equiv (-1)^k\frac{d^k}{dt^k}\left[t^k\frac{d^ku}{dt^k}\right],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{2.2}
$$

where $b_{k,j}$'s are constants whose values can be determined without any difficulty applying Leibnitz formula to $\frac{d^k}{dt^k}\left[t^k\frac{d^ku}{dt^k}\right]$. Note that

$$b_{k,0} = k! \tag{2.3}$$

Clearly, $t = 0$ for $x = \xi$ and $y = \eta$. Further, by virtue of Eqs (1.3), (2.2) and (2.3), we have

$$u\big|_{t=0} = 1 \quad , \quad \frac{du}{dt}\Big|_{t=0} = \frac{d^2u}{dt^2}\Big|_{t=0} = \cdots = \frac{d^{n-1}u}{dt^{n-1}}\Big|_{t=0} = 0 . \tag{2.4}$$

We now substitute the expressions in (2.2) into Eq (1.1). We then obtain the following $2n$-th order Fuchsian type linear differential equation:

$$
\begin{aligned}
&\frac{d^n}{dt^n}\left[t^n\frac{d^nu}{dt^n}\right] - a_1\frac{d^{n-1}}{dt^{n-1}}\left[t^{n-1}\frac{d^{n-1}u}{dt^{n-1}}\right] + \cdots\\
&\cdots + (-1)^{n-1}a_{n-1}\frac{d}{dt}\left[t\frac{du}{dt}\right] + (-1)^n a_n u = 0 .
\end{aligned} \tag{2.5}
$$

Eq (2.5) is of the form

$$t^n\frac{d^{2n}u}{dt^{2n}} + P_{2n-1}(t)\frac{d^{2n-1}u}{dt^{2n-1}} + \cdots + P_1(t)\frac{du}{dt} + a_n u = 0 \tag{2.6}$$

where $P_m(t)$'s are certain polynomials at most of order $(n-1)$ whose coefficients are determined by the numbers a_i and $b_{k,j}$'s.

Thus we have reduced the solution of the Goursat problem (1.1)–(1.3) to the solution of the differential equation (2.6) satisfying the initial conditions (2.4). This solution can be easily constructed. Indeed since $u(t)$ is analytic in the neighborhood of $t = 0$, it is of the form

$$u(t) = \sum_{m=0}^{\infty} \gamma_m t^m , \tag{2.7}$$

and, by virtue of the initial conditions (2.4), we have

$$\gamma_0 = 1 \quad , \quad \gamma_1 = \gamma_2 = \cdots = \gamma_{n-1} = 0 \,. \tag{2.8}$$

Substituting the power series (2.7) into Eq (2.6) and comparing similar terms, we obtain the following recurrence equations

$$c_{m+n}\gamma_{m+n} + c_{m+n-1}\gamma_{m+n-1} + \cdots + c_{m+1}\gamma_{m+1} + a_n\gamma_m = 0 \,, m = 0 \,, 1 \,, 2 \,, \cdots \tag{2.9}$$

where c_{m+k} $(k = 1 , 2 , \cdots , n)$ are well defined expressions in terms of the coefficients of the polynomials $P_{2n-1}(t) , \cdots , P_1(t)$. Clearly, the γ_m's are uniquely determined under the conditions (2.8)–(2.9).

Thus, Eq (1.1) has a solution satisfying the conditions (1.3). This solution is unique and is of the form (2.7), where t is defined by Eq (2.1) and γ_m's are determined by the recurrence relations (2.8)–(2.9).

3. To illustrate the above analysis, we consider the polyvibrating equation

$$\partial^2 u + au = 0 \,, \tag{3.1}$$

subject to the conditions

$$u \,|_{x=\xi} = u \,|_{y=\eta} = 1 \quad , \quad \partial u \,|_{x=\xi} = \partial u \,|_{y=\eta} = 0 \,, \tag{3.2}$$

where a is a constant. In this case we have the differential system

$$t^2 \frac{d^4 u}{dt^4} + 4t \frac{d^3 u}{dt^3} + 2 \frac{d^2 u}{dt^2} + au = 0 \quad , \quad u \,|_{t=0} = 1 \quad , \quad \left.\frac{du}{dt}\right|_{t=0} = 0 \,, \tag{3.3}$$

which correspond to Eqs (2.4) and (2.6), and the recurrence equations

$$(m+1)^2 (m+2)^2 \gamma_{m+2} + a\gamma_m = 0 \quad , \quad \gamma_0 = 1 \quad , \quad \gamma_1 = 0 \,, \tag{3.4}$$

which correspond to Eqs (2.8)–(2.9). Hence,

$$\gamma_m = (-1)^m \frac{a^m}{(2m!)^2} \,, \qquad m = 0 \,, 1 \,, 2 \,, \cdots . \tag{3.5}$$

Therefore, the solution of the Cauchy problem (3.3) is of the form

$$u(t) = \sum_{m=0}^{\infty} (-1)^m \frac{a^m}{(2m!)^2} t^{2m} \,. \tag{3.6}$$

Consequently, the function

$$u(x , y ; \xi , \eta) = \sum_{m=0}^{\infty} (-1)^m \frac{a^m}{(2m!)^2} (x-\xi)^{2m} (y-\eta)^{2m} \tag{3.7}$$

is the required solution of the Goursat problem (3.1).

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