Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A Picone integral identity for a class of fourth order elliptic differential inequalities

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.6, p. 630–641.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_50_6_630_0>

Matematica. — *A Picone integral identity for a class of fourth order elliptic differential inequalities.* Nota di D. R. DUNNINGER, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Estesa la cosiddetta *Picone integral identity* a certi operatori a derivate parziali del quart'ordine, se ne traggono teoremi di tipo sturmiano per le soluzioni di equazioni competenti a tali operatori, nonché diseguaglianze di un tipo dovuto a Wirtinger e limitazioni inferiori per gli autovalori di un parametro da cui dipendono le equazioni omogenee relative agli operatori considerati, con condizioni di nullità, alla frontiera, della soluzione e di un certo operatore su essa, lineare del second'ordine.

1. INTRODUCTION

Suppose u and v are, respectively, solutions of the equations

$$(au')' + cu = 0$$
$$(Av')' + Cv = 0$$

in the interval (x_1, x_2). If $v \neq 0$ in (x_1, x_2), then Picone's integral identity [1, p. 266]

$$\left[\frac{u}{v}(au'v - Auv')\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} [(a - A)u'^2 + (C - c)u^2]\,dx + \int_{x_1}^{x_2} A\left(u' - \frac{u}{v}v'\right)^2 dx \tag{1.1}$$

is valid.

The principal application of (1.1) has been in the proof of a more general comparison theorem than that given originally by Sturm [1, pp. 224–226].

Generalizations of (1.1) to second order elliptic differential equations have been obtained by Picone [2], Kreith [3–4], Dunninger [5] and Dunninger and Weinacht [6].

It is the purpose of this paper to present a generalization of (1.1) to a class of fourth order elliptic differential equations and a class of related elliptic differential inequalities. As an application of such an identity we shall obtain Sturmian comparison theorems under different hypotheses than those obtained in [7–11]. As further applications we shall present a Wirtinger-type inequality associated with the differential equations and lower bounds for the first eigenvalue of related eigenvalue problems.

2. THE PICONE INTEGRAL IDENTITY

The linear elliptic differential operators l and L defined by

$$lu \equiv \Delta(a\Delta u) - cu \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$
$$Lv \equiv \Delta(A\Delta v) - Cv$$

(*) Nella seduta del 18 giugno 1971.

respectively, will be considered in a bounded domain R in n–dimensional Euclidean space E^n, with a piecewise smooth boundary ∂R. The real-valued functions a and A are positive in R and of class $C^2(\overline{R})$ and the real-valued functions c and C are continuous in $\overline{R}$. The domains $\mathfrak{D}_l$ and $\mathfrak{D}_L$ of l and L, respectively, are defined to be the sets of all real-valued functions of class $C^4(\overline{R})$.

THEOREM 2.1. *If* $u \in \mathfrak{D}_l$, $v \in \mathfrak{D}_L$ *and if* $u/v \in C^2(\overline{R})$, *then*

$$\int_{\partial R} \frac{u}{v}\left(v \frac{\partial (a\Delta u)}{\partial n} - u \frac{\partial (A\Delta v)}{\partial n}\right) ds + \int_{\partial R} A \frac{\Delta v}{v}\left[\frac{u}{v}\left(v\frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right)\right] ds + \tag{2.1}$$

$$+ \int_{\partial R} \frac{1}{v}\frac{\partial u}{\partial n}(Au\,\Delta v - av\,\Delta u)\, ds =$$

$$= \int_R [(A-a)(\Delta u)^2 + (c-C)u^2]\, dx + \int_R \left[2A\frac{\Delta v}{v}\left|\operatorname{grad} u - \frac{u}{v}\operatorname{grad} v\right|^2 - \right.$$

$$\left. - A\left(\Delta u - \frac{u}{v}\Delta v\right)^2\right] dx + \int_R \frac{u}{v}(vlu - uLv)\, dx$$

where $\partial/\partial n$ *denotes the exterior normal derivative.*

Proof. Two applications of Green's second identity yield

$$\int_{\partial R}\left[u\frac{\partial(a\Delta u)}{\partial n} - a\Delta u\frac{\partial u}{\partial n}\right] ds = \int_R ulu\, dx - \int_R [a(\Delta u)^2 - cu^2]\, dx \tag{2.2}$$

and

$$\int_{\partial R}\left[A\Delta v\frac{\partial}{\partial n}\left(\frac{u^2}{v}\right) - \frac{u^2}{v}\frac{\partial(A\Delta v)}{\partial n}\right] ds = \tag{2.3}$$

$$= \int_R \left[A\Delta\, v\Delta\left(\frac{u^2}{v}\right) - Cu^2\right] dx - \int_R \frac{u^2}{v} Lv\, dx.$$

Adding these two identities together and making use of the expressions

$$\begin{aligned} &A\Delta v\Delta\left(\frac{u^2}{v}\right) = A(\Delta u)^2 + 2A\frac{\Delta v}{v}\left|\operatorname{grad} u - \frac{u}{v}\operatorname{grad} v\right|^2 - A\left(\Delta u - \frac{u}{v}\Delta v\right)^2 \\ &\frac{\partial}{\partial n}\left(\frac{u^2}{v}\right) = 2\frac{u}{v}\frac{\partial u}{\partial n} - \frac{u^2}{v^2}\frac{\partial v}{\partial n} \end{aligned} \tag{2.4}$$

we readily arrive at (2.1).

In what follows, some of the computations are simplified and slightly more general results are obtained if, instead of basing our results upon (2.1) directly, we use the following related identity:

THEOREM 2.2. *If* $u \in C^2(\bar{R})$, $v \in \mathfrak{D}_L$ *and* $u/v \in C^2(\bar{R})$, *then*

$$(2.5)\quad -\int_{\partial R} \frac{u^2}{v} \frac{\partial (A\Delta v)}{\partial n}\, ds + \int_{\partial R} A \frac{\Delta v}{v} \left[\frac{u}{v}\left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right)\right] ds + \int_{\partial R} A \frac{\Delta v}{v}\, u \frac{\partial u}{\partial n}\, ds =$$

$$= \int_R [A(\Delta u)^2 - Cu^2]\, dx +$$

$$+ \int_R \left[2A \frac{\Delta v}{v} \left|\operatorname{grad} u - \frac{u}{v} \operatorname{grad} v\right|^2 - A\left(\Delta u - \frac{u}{v}\Delta v\right)^2\right] dx - \int_R \frac{u^2}{v} Lv\, dx .$$

Proof. The identity (2.5) readily follows by either setting $a = c = 0$ in (2.1) or by substituting (2.4) into (2.3).

3. STURMIAN COMPARISON THEOREMS

We begin with the following preliminary result.

LEMMA 3.1. *If there exists a nontrivial real-valued function* $u \in C^2(\bar{R})$ *which satisfies*

$$u = 0 \quad on \quad \partial R$$

$$M[u] \equiv \int_R [A(\Delta u)^2 - Cu^2]\, dx \leq 0$$

then there does not exist a $v \in \mathfrak{D}_L$ *which satisfies*

$$(3.1)\quad \begin{aligned} Lv &\geq 0 \quad in \quad R \\ v &> 0 \quad on \quad \partial R \\ \Delta v &< 0 \quad in \quad R . \end{aligned}$$

Proof: Suppose a solution v of (3.1) exists. Since $\Delta v < 0$ in R and $v > 0$ on ∂R it follows (using the maximum principle) that $v > 0$ in $\bar{R}$. Consequently (2.5) is valid and readily implies, in view of the above hypotheses, that

$$0 \geq M[u] - \int_R \frac{u^2}{v} Lv\, dx = -\int_R \left[2A \frac{\Delta v}{v} \left|\operatorname{grad} u - \frac{u}{v} \operatorname{grad} v\right|^2 - \right.$$

$$\left. - A\left(\Delta u - \frac{u}{v}\Delta v\right)^2\right] dx \geq 0 .$$

As a consequence, $\operatorname{grad} u - \frac{u}{v} \operatorname{grad} v = 0$ and $\Delta u - \frac{u}{v}\Delta v = 0$ in R and therefore, $u/v = k$ in $\bar{R}$ for some nonzero constant k. However, this latter condition cannot hold since $u = 0$ on ∂R whereas $v \neq 0$ on ∂R. Hence no such solution v can exist.

THEOREM 3.1. *Suppose $\partial R \in C^2$. If there exists a nontrivial real-valued function $u \in C^2(\bar{R})$ which satisfies*

$$u = \frac{\partial u}{\partial n} = 0 \qquad on \quad \partial R \tag{3.2}$$

$$M[u] \leq 0 \tag{3.3}$$

then every $v \in \mathfrak{D}_L$ which satisfies

$$\begin{aligned} Lv &\geq 0 && in \quad R \\ v &> 0 && for\ some \quad x \in R \\ \Delta v &< 0 && in \quad R \end{aligned}$$

must vanish at some point in R unless u is a constant multiple of v. Moreover, if $M[u] < 0$, then v must vanish at some point in R.

Proof: The function v belongs to one of two classes: either $v < 0$ for some $x \in \partial R$, or $v \geq 0$ for all $x \in \partial R$. If $v < 0$ for some $x \in \partial R$, then together with the assumption $v > 0$ for some $x \in R$, we have that v must vanish at some point in R. (Note that this result is independent of any hypotheses concerning u and any differential inequalities satisfied by u or v). On the other hand, if $v \geq 0$ on ∂R, it follows from Lemma 3.1 that $v = 0$ for some $x \in \partial R$. This, together with the assumption $\Delta v < 0$ in R, implies (using the maximum principle) that $v > 0$ in R.

Let $\overset{\circ}{H}_2(R)$ denote the Sobolev space, which is the closure in the norm $\|\cdot\|_2$ defined by

$$\|u\|_2^2 = \int_R |u|^2 \, dx + \sum_{i=1}^{n} \int_R |D_i^2 u|^2 \, dx \quad , \quad D_i^2 = \frac{\partial^2}{\partial x_i^2} \tag{3.4}$$

of the class $C_0^\infty(R)$ of infinitely differentiable functions with compact support in R.

Since $\partial R \in C^2$ and u satisfies the boundary conditions in (3.2), it is known [12, p. 131] that $u \in \overset{\circ}{H}_2(R)$. Let $\{u_m\}$ denote a sequence of $C_0^\infty(R)$ functions converging to u in the norm (3.4). Since u_m vanishes in a neighborhood of ∂R and $v > 0$ in R, it follows that u_m/v is at least of class $C^2(\bar{R})$ and therefore the identity (2.5) is valid. Consequently, under the above hypotheses we obtain from (2.5) that

$$M[u_m] = -\int_R \left[2A \frac{\Delta v}{v} \left| \operatorname{grad} u_m - \frac{u_m}{v} \operatorname{grad} v \right|^2 - A\left(\Delta u_m - \frac{u_m}{v}\Delta v\right)^2 \right] dx + \int_R \frac{u_m^2}{v} Lv \, dx \geq 0 . \tag{3.5}$$

Since A and C are bounded, there exists a positive constant K_1 such that

$$|M[u_m] - M[u]| \leq K_1 \int_R |\Delta u_m \Delta(u_m - u) + \Delta u \Delta(u_m - u)| \, dx$$

$$+ K_1 \int_R |u_m(u_m - u) + u(u_m - u)| \, dx$$

which yields, upon applying the Schwarz inequality, the estimate

$$|M[u_m] - M[u]| \leq K_1(n^2 + 1)(\|u_m\|_2 + \|u\|_2)\|u_m - u\|_2 . \tag{3.6}$$

Since $\|u_m - u\|_2 \to 0$ as $m \to \infty$, we obtain from (3.6) that $M[u_m] \to M[u]$ as $m \to \infty$ and therefore from (3.5), $M[u] \geq 0$. If $M[u] > 0$, we obtain a contradiction in view of (3.3), and therefore $M[u] = 0$.

Let B denote a ball with $\overline{B} \subset R$ and define

$$Q_B[u_m] \equiv -\int_B \left[2A \frac{\Delta v}{v} \left|\operatorname{grad} u_m - \frac{u_m}{v} \operatorname{grad} v\right|^2 - A\left(\Delta u_m - \frac{u_m}{v}\Delta v\right)^2\right] dx .$$

Setting $w = u/v$ and $w_m = u_m/v$ and using the first expression in (2.4) we have

$$Q_B[u_m] = \int_B [A(\Delta u_m)^2 - A\Delta v \Delta(u_m w_m)] \, dx .$$

Since Δv is bounded, there exists a positive constant K_2 such that

$$|Q_B[u_m] - Q_B[u]| \leq K_2 \int_B |\Delta u_m \Delta(u_m - u) + \Delta u \, \Delta(u_m - u)| \, dx$$

$$+ K_2 \int_B |\Delta[u_m(w_m - w)] + \Delta[w(u_m - u)]| \, dx$$

which yields, upon applying the Schwarz inequality, the estimate

$$|Q_B[u_m] - Q_B[u]| \leq K_2 n^2 (\|u_m\|_{2,B} + \|u\|_{2,B})\|u_m - u\|_{2,B}$$

$$+ K_2(\|u_m\|_{2,B}\|w_n - w\|_{2,B} + \|w\|_{2,B}\|u_m - u\|_{2,B})$$

where the subscript B indicates that the integrals occurring in the norm (3.4) are to be evaluated over B only. Since $v \neq 0$ in $\overline{B}$, it follows from the definitions of w_m and w that $\|w_m - w\|_{2,B} \to 0$ as $\|u_m - u\|_2 \to 0$ and consequently, $Q_B[u_m] \to Q_B[u]$ as $m \to \infty$. Finally since

$$0 \leq Q_B[u_m] \leq M[u_m]$$

it now follows that $Q_B[u] = 0$, and therefore $\operatorname{grad} u - \frac{u}{v} \operatorname{grad} v = 0$ and $\Delta u - \frac{u}{v}\Delta v = 0$ in B. Since B is arbitrary, $u/v = k$ in $\overline{R}$, for some nonzero constant k.

To prove the second statement, we note that if $M[u] < 0$, then the same proof as above leads immediately to the contradiction $M[u] \geq 0$. This completes the proof.

Remark. The technique used in the proof of Theorem 3.1, namely, the introduction of an appropriate Sobolev space, was motivated by some recent works of Allegretto [13] and Swanson [14], in which this device was used in connection with Sturmian theorems for second order elliptic equations and systems.

Our main result now follows.

THEOREM 3.2. *Suppose* $\partial R \in C^2$. *If there exists a nontrivial* $u \in \mathfrak{D}_l$ *which satisfies*

$$ulu \leq 0 \quad \textit{in} \quad R \tag{3.7}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \textit{on} \quad \partial R \tag{3.8}$$

$$V[u] \equiv \int_R [(a - A)(\Delta u)^2 + (C - c) u^2] \, dx \geq 0$$

then every $v \in \mathfrak{D}_L$ *which satisfies*

$$Lv \geq 0 \quad \textit{in} \quad R$$

$$v > 0 \quad \textit{for some} \quad x \in R$$

$$\Delta v < 0 \quad \textit{in} \quad R$$

must vanish at some point in R *unless* u *is a constant multiple of* v. *Moreover, if* $V[u] > 0$, *then* v *must vanish at some point in* R.

Proof. The hypothesis $V[u] \geq 0$ is equivalent to

$$M[u] \leq \int_R [a(\Delta u)^2 - cu^2] \, dx \, . \tag{3.9}$$

Since u satisfies (3.7) and (3.8) it follows from (2.2) that the right side of (3.9) vanishes and hence the condition $M[u] \leq 0$ of Theorem 3.1 is fulfilled. The second statement is proved similarly.

Several consequences of Theorem 3.2 are now considered.

COROLLARY 3.1. *Suppose* $\partial R \in C^2$. *If there exists a nontrivial* $u \in \mathfrak{D}_l$ *which satisfies*

$$ulu \leq 0 \quad \textit{in} \quad R$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \textit{on} \quad \partial R$$

$$V[u] \geq 0$$

and if $v \in \mathfrak{D}_L$ *satisfies*

$$\begin{aligned} &Lv \geq 0 && \text{in } R \\ &v > 0 && \text{for some } x \in R \\ &\Delta v < 0 && \text{for some } x \in R \end{aligned}$$

then either v *or* Δv *must vanish at some point in* R *unless* u *is a constant multiple of* v. *Moreover, if* $V[u] > 0$, *then either* v *or* Δv *must vanish at some point in* R.

COROLLARY 3.2. *Suppose* $\partial R \in C^2$. *If* $C \geq 0$ $(C \not\equiv 0)$ *in* R *and if there exists a nontrivial* $u \in \mathfrak{D}_l$ *which satisfies*

$$\begin{aligned} &ulu \leq 0 && \text{in } R \\ &u = \frac{\partial u}{\partial n} = 0 && \text{on } \partial R \\ &V[u] \geq 0 \end{aligned}$$

then every $v \in \mathfrak{D}_L$ *which satisfies*

$$\begin{aligned} &Lv \geq 0 && \text{in } R \\ &v > 0 && \text{for some } x \in R \\ &\Delta v = 0 && \text{on } \partial R \end{aligned} \tag{3.10}$$

must vanish at some point in R *unless* u *is a constant multiple of* v. *Moreover, if* $V[u] > 0$, *then* v *must vanish at some point in* R.

Proof. If $v \neq 0$ in R, then it follows from (3.10) that $v > 0$ in R. Defining $w = A\Delta v$, it follows that w satisfies the system

$$\begin{aligned} &\Delta w \geq 0 \; (\Delta w \not\equiv 0) && \text{in } R \\ &w = 0 && \text{on } \partial R \end{aligned}$$

and therefore (using the maximum principle) $w < 0$ and hence $\Delta v < 0$ in R. Theorem 3.2 now implies that u is a constant multiple of v. The second statement is proved similarly.

The next result is concerned with replacing the integral inequality $V[u] \geq 0$ by pointwise inequalities and in relaxing the assumption $A > 0$ in R.

COROLLARY 3.3. *Suppose* $\partial R \in C^2$. *Suppose* $a \geq A \geq 0$, $C \geq c$ in R *and* $u \in \mathfrak{D}_l$, $u \not\equiv 0$ *in any open subset of* R. *If either*

$$C \not\equiv c \quad \text{in } R$$

and u *satisfies*

$$\begin{aligned} &ulu \leq 0 && \text{in } R \\ &u = \frac{\partial u}{\partial n} = 0 && \text{on } \partial R \end{aligned}$$

or

$$a > A \quad \textit{and} \quad c \not\equiv 0 \quad \textit{for the same} \quad x \in R$$

and u satisfies

$$lu = 0 \qquad in\ R$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad on\ \partial R$$

then every $v \in \mathfrak{D}_L$ *which satisfies*

$$\begin{aligned} Lv &\geq 0 && in\ R \\ v &> 0 && for\ some\ x \in R \\ \Delta v &< 0 && in\ R \end{aligned}$$

must vanish at some point in R.

Proof: Clearly $V[u] \geq 0$ is implied by the pointwise conditions $a \geq A$, $C \geq c$ in R. Moreover, it is readily seen that the second conclusion in Theorems 3.1 and 3.2 remains valid if $A \geq 0$ in R. Since $u \not\equiv 0$ in any open subset of R, it follows that $V[u] > 0$ when $C \not\equiv c$ in R. In the case $a > A$ and $c \not\equiv 0$ for the same $x \in R$, it follows that $V[u] = 0$ only if $\Delta u \equiv 0$ in some nonempty open subset $S \subset R$. Since $c \not\equiv 0$ at some $x_0 \in S$, it is easily verified that the differential equation $lu = 0$ is not satisfied at x_0. Consequently, $V[u] > 0$ in this case also, and the conclusions follow from Theorem 3.2.

The previous results admit a generalization in which the solutions of $ulu \leq 0$ in R satisfy the mixed boundary conditions

$$u = 0 \quad , \quad \Delta u + \alpha \frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad \partial R$$

where $0 \leq \alpha \leq +\infty \left(\alpha = +\infty \text{ denotes the boundary condition } \frac{\partial u}{\partial n} = 0\right)$. However, the results are " weak " in the sense that the conclusions with respect to v apply to $\overline{R}$ rather than R. For the weak theorems, ∂R is required only to be piecewise smooth.

In this direction, we shall be content to state a typical result, leaving its proof to the reader.

THEOREM 3.3. *If there exists a nontrivial* $u \in \mathfrak{D}_l$ *which satisfies*

$$ulu \leq 0 \qquad in\ R$$

$$u = 0 \quad , \quad \Delta u + \alpha \frac{\partial u}{\partial n} = 0 \qquad on\ \partial R, \qquad (0 \leq \alpha \leq +\infty)$$

$$V[u] \geq 0$$

and if $v \in \mathfrak{D}_L$ *satisfies*

$$\begin{aligned} Lv &\geq 0 && in\ R \\ v &> 0 && for\ some\ x \in R \\ \Delta v &< 0 && for\ some\ x \in R \end{aligned} \tag{3.11}$$

then either v must vanish at some point in $\overline{R}$, *or* Δv *must vanish at some point in* R.

4. Wirtinger inequalities

By a mere reinterpretation of Theorem 3.1 and a simple application of the identity (2.2) we readily obtain a Wirtinger inequality analogous to the type considered by Beesack [15] for second order and fourth order ordinary differential equations.

THEOREM 4.1. *Suppose* $\partial R \in C^2$. *If there exists a* $v \in \mathfrak{D}_L$ *which satisfies*

$$
\begin{aligned}
Lv &= 0 \quad \textit{in} \quad R \\
v &> 0 \quad \textit{in} \quad R \\
\Delta v &< 0 \quad \textit{in} \quad R
\end{aligned} \tag{4.1}
$$

Then every nontrivial real-valued function $u \in C^2(\bar{R})$ *which satisfies*

$$u = \frac{\partial u}{\partial n} = 0 \qquad on \quad \partial R$$

also satisfies the inequality

$$\int_R A\,(\Delta u)^2\,dx \geq \int_R Cu^2\,dx \tag{4.2}$$

with equality if and only if u is a constant multiple of v.

In [15] Beesack made explicit use of an associated Riccati transform in order to establish inequalities of the type (4.2). Although the Riccati transform is not used explicitly in our work, it is nevertheless implicitly evident and we refer the reader to [16] where this idea is carried out for fourth order elliptic differential equations.

Various other Wirtinger inequalities are possible by varying the condition on v in (4.1) or by varying the boundary conditions on u (see [15], Lemma 2.1) however, we shall not pursue this matter here.

5. Lower bounds for eigenvalues

As a final application we shall consider eigenvalue problems of the form

$$
\begin{aligned}
&lu = \lambda u \qquad in \quad R \\
&u = 0 \quad , \quad \Delta u + \alpha \frac{\partial u}{\partial n} = 0 \qquad on \quad \partial R
\end{aligned} \tag{5.1}
$$

where $0 \leq \alpha \leq +\infty$.

In particular, we shall derive lower bounds analogous to those derived by Barta [17] and Protter and Weinberger [18] for second order elliptic equations.

THEOREM 5.1. *Let* λ *be the lowest eigenvalue with corresponding eigenfunction* $u \in \mathfrak{D}_l$ *of the problem* (5.1). *If* v *is any real-valued function of class* $C^4(\overline{R})$ *which satisfies*

$$v > 0 \qquad in \quad \overline{R}$$
$$\Delta v \leq 0 \qquad in \quad R$$

and if

$$V[u] \geq 0$$

then

$$\lambda \geq \inf_{x \in R} \left(\frac{Lv}{v}\right). \tag{5.2}$$

Proof: In view of the above hypotheses, (2.1) implies that

$$\lambda \int_R u^2 \, dx - \int_R u^2 \left(\frac{Lv}{v}\right) dx =$$

$$= V[u] - \int_R \left[2 A \frac{\Delta v}{v} \left|\operatorname{grad} u - \frac{u}{v} \operatorname{grad} v\right|^2 - A \left(\Delta u - \frac{u}{v} \Delta v\right)^2\right] dx +$$

$$+ \int_{\partial R} \alpha a \left(\frac{\partial u}{\partial n}\right)^2 ds \geq 0$$

from which (5.2) is a simple consequence.

6. CONCLUDING REMARKS

The techniques of this paper allow generalizations to more general differential operators, e.g.,

(*i*) the non-self-adjoint operators l_1 and L_1 defined, respectively, by

$$l_1 u \equiv \Delta(a \Delta u) + 2 b \Delta u - cu$$
$$L_1 v \equiv \Delta(A \Delta v) + 2 B \Delta v - Cv.$$

Here the associated Picone integral identity is

$$\int_{\partial R} \frac{u}{v} \left(v \frac{\partial (a \Delta u)}{\partial n} - u \frac{\partial (A \Delta v)}{\partial n}\right) ds + \int_{\partial R} A \frac{\Delta v}{v} \left[\frac{u}{v} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right)\right] ds +$$

$$+ \int_{\partial R} \frac{1}{v} \frac{\partial u}{\partial n} (A u \Delta u - a v \Delta u) \, ds =$$

$$= \int_R [(A - a)(\Delta u)^2 + 2 (B - b) u \Delta u + (G - C + c) u^2] \, dx -$$

$$- \int_R \left[A \left(\Delta u - \frac{u}{v} \Delta v\right)^2 + 2 B u \left(\Delta u - \frac{u}{v} \Delta v\right) + G u^2\right] dx +$$

$$+ \int_R 2 A \frac{\Delta v}{v} \left|\operatorname{grad} u - \frac{u}{v} \operatorname{grad} v\right|^2 dx + \int_R \frac{u}{v} (v l_1 u - u L_1 v) \, dx$$

where the continuous function G is chosen so that the following quadratic form in $\Delta u - \frac{u}{v}\Delta v$ and u:

$$A\left(\Delta u - \frac{u}{v}\Delta v\right)^2 + 2\,Bu\left(\Delta u - \frac{u}{v}\Delta v\right) + Gu^2$$

is positive definite. A necessary and sufficient condition for this to be the case is $A > 0$, $AG - B^2 > 0$.

(*ii*) the operators l_2 and L_2 defined, respectively, by

$$l_2\,u \equiv \Sigma D_{ij}\,(a^{ij}\,a^{kl}\,D_{kl}\,u) - cu$$
$$L_2\,v \equiv \Sigma D_{ij}\,(A^{ij}\,A^{kl}\,D_{kl}\,v) - Cv$$

where the matrixes (a^{ij}) and (A^{ij}) are symmetric and positive definite. Here the associated Picone integral identity is

$$\int_{\partial R} \frac{u}{v}\,[v\Sigma D_j\,(a^{ij}\,a^{kl}\,D_{kl}\,u)\,n_i - u\Sigma D_j\,(A^{ij}\,A^{kl}\,D_{kl}\,v)\,n_i]\,ds +$$
$$+ \int_{\partial R}\left[\Sigma A^{ij}\,A^{kl}\,D_{kl}\,vD_i\left(\frac{u^2}{v}\right)n_i\right]ds - \int_{\partial R}[\Sigma a^{ij}\,a^{kl}\,D_{kl}\,uD_j\,un_i]\,ds =$$
$$= \int_R \Sigma\,[(A^{ij})^2 - (a^{ij})^2]\,(D_{ij})^2\,dx +$$
$$+ \int_R \frac{2}{v}\,\Sigma A^{kl}\,D_{kl}\,v\Sigma A^{ij}\left(D_i\,u - \frac{u}{v}\,D_i\,v\right)\left(D_j\,u - \frac{u}{v}\,D_j\,v\right)dx -$$
$$- \int_R\left[\Sigma A^{ij}\left(D_{ij}\,u - \frac{u}{v}\,D_{ij}\,v\right)\right]^2 dx + \int_R \frac{u}{v}\,(vl_2\,u - uL_2\,v)\,dx$$

where $n = (n_1, \cdots, n_n)$ is the unit exterior normal vector to the boundary ∂R.

Acknowledgement. I would like to thank Professors J. B. Diaz and J. R. McLaughlin for some helpful suggestions during the preparation of this paper.

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