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Admissible sets and Kuratomski's number α

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Topologia. — *Admissible sets and Kuratowski's number* α (*). Nota di CARLO FRANCHETTI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Usando il concetto di insieme ammissibile si dimostra la seguente proprietà del numero α di Kuratowski:

$$\alpha(cl\,A) = \alpha(A),$$

dove cl è la chiusura in una conveniente topologia debole di uno spazio normato.

1. METRIC SPACES

Let (X, d) be a metric space. If A is a bounded subset of X we denote by $\delta(A)$ the diameter of A, that is the number

$$\delta(A) = \sup_{a,a' \in A} d(a, a').$$

If $a \in X$, $r > 0$, $A \subset X$ we denote by $B(a, r)$ $[B(A, r)]$ the closed ball with center in a [A] and radius r, i.e. the set

$$B(a, r) = \{y \in X : d(y, a) \leq r\} \qquad [B(A, r) = \{y \in X : d(y, A) \leq r\}].$$

Note that if A is bounded, then $\delta[B(A, r)] \leq \delta(A) + 2r$.

Let again A be a bounded subset of X, we pose

$$f_A(x) = \sup_{a \in A} d(a, x).$$

f_A is a real nonnegative functional defined on X which satisfies

$$|f_A(x) - f_A(y)| \leq d(x, y), \qquad \forall x, y \in X;$$

i.e. f_A is nonexpansive. A point $a_0 \in A$ such that $d(x, a_0) = f_A(x)$ is called a farthest point to x in A (see for ex. [1]).

A bounded set $A \subset X$ is called admissible if it is the intersection of a family of closed balls. Obviously the intersection of any family of admissible sets is an admissible set. So if B is a bounded set we can define a new set $B_1 \supset B$ as the smallest admissible set which contains B, in other words B_1 is the intersection of all closed balls which contain B. Note that one can generalize the above definition to unbounded sets. In fact call a set $V \subset X$ admissible (in the generalized sense) if

i) $W \subset V$, W bounded $\Rightarrow W_1 \subset V$.

It is easily seen that *i*) is equivalent to

ii) $\forall B \subset X$, B admissible, $B \cap V$ is admissible.

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We can define the set V_1 as above (note that X is admissible in the generalized sense).

It is easy to see that if A is bounded we have

$$A_1 = \{y \in X : d(y, x) \leq f_A(x), \forall x \in X\} = \bigcap_{x \in X} B(x, f_A(x)).$$

We remark that $f_A(x) \geq \delta(A)/2$; so A_1 may also be considered as the intersection of all closed balls which contain A and with radius $r \geq \delta(A)/2$. From the definition of A_1 it follows immediately that if a_0 is a farthest point from x in A, then a_0 is also a farthest point from x in A_1.

PROPOSITION 1. *If* X *is a separable metric space, every bounded admissible set is the intersection of a countable family of closed balls.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be an everywhere dense sequence in X, $A \subset X$ a bounded admissible set: $A = \{y \in X : d(y, x) \leq f_A(x), \forall x \in X\}$. Consider the set $A' = \{y \in X : d(y, x_n) \leq f_A(x_n), \forall n\}$; A' is the intersection of a countable family of closed balls and $A \subset A'$. We must prove that $A' = A$. Suppose that $y \in A'$, then $d(y, x_n) \leq f_A(x_n)$, $\forall n$. If x is any point in X, there exists a subsequence of $\{x_n\}$ which converges to x. Continuity of f_A implies then that $d(y, x) \leq f_A(x)$.

Suppose that A is a bounded subset of X, let us now consider more generally the set A_c: $A_c = \{y \in X : d(y, x) \leq cf_A(x), \forall x \in X\}$, where $c \geq 1$.

For $c = 1$ we get the above defined set A_1; for every $c > 1$, A_c is an admissible set which contains A_1. A_c is also the intersection of all closed balls which contain A and with radius $r \geq c\delta(A)/2$.

THEOREM 1. *If* A *is a bounded subset of* X, $c \geq 1$ *then*

$$\delta(A_c) \leq c\delta(A), \quad \textit{in particular}$$

$$\delta(A_1) = \delta(A).$$

Proof. Recall that $A_c = \{y \in X : d(z, y) \leq cf_A(z), \forall z \in X\}$. From the definition of A_c we get: $f_{A_c}(z) \leq cf_A(z)$. Hence

$$\sup_{\substack{z \in A \\ y \in A_c}} d(y, z) \leq c\delta(A). \tag{1}$$

So, in order to prove the assertion, it is enough to consider pairs $y_1, y_2 \in A_c$, $y_1, y_2 \notin A$. Suppose that $\delta(A_c) > c\delta(A)$; then there exist $y_1, y_2 \in A_c \setminus A$ such that $r = d(y_1, y_2) > c\delta(A)$. If s is such that $c\delta(A) < s < r$ put $G = B(y_1, s) \cap A_c$. Since $y_2 \notin G$, G is properly included in A_c and so cannot contain all A. (In fact A_c is the intersection of all closed balls which contain A and with radius $\geq c\delta(A)/2$, since $B(y_1, s)$ has radius $> c\delta(A)$, if $B \supset A$ it must be $G \supset A_c$ which is a contradiction). Let $b \in A$, $b \notin B(y_1, s)$; we have $d(b, y_1) > s > c\delta(A)$ which is a contradiction with (1).

Remark 1. Let A be a bounded set in X. We recall the definition of Kuratowski's number $\alpha(A)$ (see [2], pag. 318)

$\alpha(A) = \inf \varepsilon$ $\varepsilon > 0$ such that A can be covered by a finite collection of sets $\{B_k\}_{k=1}^{n}$ with $\delta(B_k) < \varepsilon$, $k = 1, \cdots, n$.

Because of theorem 1 we can choose the sets of the covering in the family of admissible sets without changing the value of $\alpha(A)$.

2. Normed spaces

Suppose now that X is a normed space. Theorem 1 may be formulated more precisely:

THEOREM 2. *If* A *is a bounded subset of* X, $c \geq 1$ *then*

$$\delta(A_c) = c\delta(A). \tag{2}$$

Proof. We can suppose $\delta(A) > 0$, $c > 1$ and prove that $\delta(A_c) \geq c\delta(A)$.

We begin to remark that in a normed space if C is bounded then $\delta[B(C, r)] = \delta(C) + 2r$. If

$$A_c \supset B(A, (c-1)\delta(A)/2) \tag{3}$$

holds, then

$$\delta(A_c) \geq \delta[B(A, (c-1)(A)/2)] = \delta(A) + (c-1)\delta(A) = c\delta(A)$$

and so also (3) holds.

So we prove (3): let $y \in B(A, (c-1)\delta(A)/2)$; $\forall x \in X$ we have: $d(y, x) \leq d(y, a) + d(a, x)$, $\forall a \in A$. $\forall \varepsilon > 0$ we can choose $a_0 \in A$ such that $d(y, a_0) < \frac{(c-1)\delta(A)}{2} + \varepsilon$ and so:

$$d(y, x) < \frac{(c-1)\delta(A)}{2} + d(a_0, x) + \varepsilon \leq \frac{(c-1)\delta(A)}{2} + f_A(x) + \varepsilon.$$

Since $1/f_A(x) \leq 2/\delta(A)$ we get $d(y, x) < \left[\frac{(c-1)\delta(A)}{2f_A(x)} + 1\right] f_A(x) + \varepsilon \leq cf_A(x) + \varepsilon$, ε being arbitrary, we have $d(y, x) \leq cf_A(x)\, \forall x$, i.e. $y \in A_c$.

COROLLARY 1. *If* A *is bounded* $c > 1$, *then*

$$A_c = B_1, \quad \text{where} \quad B = B(A, (c-1)\delta(A)/2).$$

Proof. $B \subset A_c \Rightarrow B_1 \subset A_c$ since A_c is admissible. Otherwise every ball which contains B_1 has radius $\geq c\delta(A)/2$, hence $B_1 \supset A_c$.

COROLLARY 2 (see [3]). A *bounded* $\Rightarrow \delta(A) = \delta(\overline{co}A)$.

Proof. In fact $\overline{co}A \subset A_1$ since A_1 is closed and convex.

Remark 2. Theorem 2 is the best possible as we shall see.

Let X be a set, $\mathfrak{F}$ a family of subsets of X such that:

i) $X \in \mathfrak{F}$; *ii*) $\mathfrak{F}$ is closed under intersection; then the map φ defined on subsets of X, $\varphi : A \to \bigcap_{\substack{F \in \mathfrak{F} \\ F \supset A}} F$ is called a C–closure (see [4]). Let $\mathfrak{F}$ consist of a collection of closed convex sets in a normed space X, then if:

$$\delta(\varphi(A)) = \delta(A) \quad \text{holds} \quad \forall A \subset X, \ A \text{ bounded},$$

$\mathfrak{F}$ contains closed balls hence $\mathfrak{F}$ contains admissible sets.

Indeed suppose that B is a closed ball, $B \notin \mathfrak{F}$, then $\varphi(B)$ contains properly B and so $\delta(\varphi(B)) > \delta(B)$.

3. Norm determining topologies

We shall give now some applications of remark 1.

Let X be a normed space, Φ a subspace of X^* (the norm dual of X).

We say that the Φ–topology of X (which we denote by τ) is norm determining (n.d.) with characteristic ν $(0 < \nu \leq 1)$ if:

$$\forall x \in X \quad \sup_{\substack{\varphi \in \Phi \\ \|\varphi\|=1}} |\varphi(x)| \geq \nu \|x\|, \text{ and } \nu \text{ is the greatest possible constant.}$$

The following are known results (see [5]):

i) if τ is n.d., $\nu = 1$ then every closed ball B is τ–closed $(B = \bar{B}^{\tau})$ and so every admissible set is τ–closed. If $0 < \nu < 1$ then $\forall x \in X$ $\overline{B(x, r)}^{\tau} \subset B\left(x, \frac{1}{\nu} r\right)$. Consequently:

ii) $\bar{A}^{\tau} \subset A_{1/\nu}$.

PROPOSITION 2. *If* A *is bounded,* τ *is n.d., then*:

$$\delta(\bar{A}^{\tau}) \leq \frac{1}{\nu}\delta(A), \quad \text{in particular} \quad \delta(\bar{A}^{\tau}) = \delta(A) \quad \text{if} \quad \nu = 1.$$

Proof. Follows from *ii*) and theorem 2.

THEOREM 3. *Let* τ *be a n.d. topology on* X *with characteristic* ν, A *a bounded subset of* X, *then*:

$$\alpha(A) \leq \alpha(\bar{A}^{\tau}) \leq \frac{1}{\nu}\alpha(A), \quad \textit{in particular if} \quad \nu = 1$$

$$\alpha(A) = \alpha(\bar{A}^{\tau}).$$

Proof. Put $\alpha(A) = \alpha$, $\varepsilon > 0$ arbitrary; there exists a finite family $\{C_i\}_{i=1}^{n}$ of admissible sets with $\delta(C_i) < \alpha + \varepsilon$ such that $A \subset C_1 \cup \cdots \cup C_n$; hence

$$\bar{A}^{\tau} \subset \overline{C_1 \cup \cdots \cup C_n}^{\tau} = \bar{C}_1^{\tau} \cup \cdots \cup \bar{C}_n^{\tau} \quad ; \quad \delta(\bar{C}_i^{\tau}) \leq \frac{1}{\nu}\delta(C_i) < \frac{1}{\nu}\alpha + \frac{\varepsilon}{\nu}.$$

Therefore

$$\alpha(\bar{A}^{\tau}) \leq \frac{1}{\nu}\alpha = \frac{1}{\nu}\alpha(A).$$

COROLLARY 3. *If* A *is bounded, then*

$$\alpha(\overline{co}^{\tau} A) \leq \frac{1}{\nu}\alpha(A). \tag{4}$$

Proof. It is enough to use a result of Darbo (see [3]) which asserts that

$$\alpha(A) = \alpha(\overline{co}\, A) . \tag{5}$$

Suppose now that A is τ–compact and convex, using Krein–Milman theorem (see [6] pag. 439–440) we get $A = \overline{co}^{\tau}\, \mathrm{ext}\, A$, where ext A is the set of extremal points of A. So we have:

COROLLARY 4. *If* A *is τ–compact and convex, then*

$$\alpha(A) \leq \frac{1}{\nu}\, \alpha(\mathrm{ext}\, A) .$$

Proof. Use (4).

We give an example. If X is an adjoint space, the w^*–topology on X is n.d. with characteristic $\nu = 1$. Since the closed unit ball U is w^*–compact, we get $\alpha(U) = \alpha(\mathrm{ext}\, U)$. It is known that $\alpha(U) = 2$ in every infinite dimensional normed space (see [7]). So we have:

COROLLARY 5. *If the closed unit ball* U *of an infinite dimensional normed space is τ–compact for a n.d. topology τ with characteristic $\nu = 1$, then*

$$\alpha(\mathrm{ext}\, U) = 2 . \tag{6}$$

Remark that if U is weakly compact then (6) follows from (5), but in the general case we must use (4).

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