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Special Functions of Mathematical Physics With Several Variables. I: Bessel Functions

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Analisi matematica. — *Special Functions of Mathematical Physics With Several Variables.* I: *Bessel Functions* (*). Nota di MEHMET NAMIK OĞUZTÖRELI, presentata (**) dal Socio M. PICONE.

RIASSUNTO. — In questo articolo, è studiata una generalizzazione di funzioni di Bessel di prima specie per più variabili.

1. INTRODUCTION

In a series of papers, D. Mangeron and the author have investigated certain extensions of some of the special functions of the Mathematical Physics to several variables (cf. [1]–[5]). In this paper we continue our investigation concerning generalized Bessel functions of the first kind.

It is well known that the Bessel functions of order ν of the first kind

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}, \tag{1.1}$$

can be characterized as the solution of Bessel's differential equation

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \nu^2)\,u = 0, \tag{1.2}$$

or as the solution of the differential recurrence relations

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x) \quad , \quad \frac{d}{dx}[x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x), \tag{1.3}$$

normalized by the condition

$$\left(\frac{x}{2}\right)^{-\nu} J_\nu(x)\Big|_{x=0} = \frac{1}{\Gamma(\nu+1)}, \tag{1.4}$$

where $\Gamma(x)$ is the gamma function.

In this paper we investigate analytic functions

$$J_{\nu_1,\nu_2,\cdots,\nu_n} = J_{\nu_1,\nu_2,\cdots,\nu_n}(x_1, x_2, \cdots, x_n)$$

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satisfying the functional equations

$$(1.5)\quad \begin{cases} \partial_n \left(x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n} J_{\nu_1,\nu_2,\cdots,\nu_n}\right) = x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n} J_{\nu_1-1,\nu_2-1,\cdots,\nu_n-1} \\ \partial_n \left(x_1^{-\nu_1} x_2^{-\nu_2} \cdots x_n^{-\nu_n} J_{\nu_1,\nu_2,\cdots,\nu_n}\right) = -x_1^{-\nu_1} x_2^{-\nu_2} \cdots x_n^{-\nu_n} J_{\nu_1+1,\nu_2+1,\cdots,\nu_n+1} \end{cases}$$

and normalized by the conditions

$$(1.6)\qquad \{x_1^{-\nu_1} x_2^{-\nu_2} \cdots x_n^{-\nu_n} J_{\nu_1,\nu_2,\cdots,\nu_n}\}\big|_{x_k=0} = \frac{1}{\prod_{j=1}^{n} 2^{\nu_j}\,\Gamma(\nu_j+1)} \qquad (k=1,2,\cdots,n),$$

where ∂_n is M. Picone's " total derivative operator " [6],

$$(1.7)\qquad \partial_n = \frac{\partial^n}{\partial x_1\,\partial x_2 \cdots \partial x_n}\,.$$

Note that the parameters $\nu_1, \nu_2, \cdots, \nu_n$ and the variables $x_1, x_2, \cdots, x_n$ can be arbitrarily complex. Clearly, for $n = 1$ Eqs. (1.5) and (1.6) reduce to Eqs (1.3) and (1.4), respectively.

We can easily verify that any solution $J = J_{\nu_1,\nu_2,\cdots,\nu_n}$ of the functional equations (1.5) satisfies necessarily the Mangeron equation

$$(1.8)\qquad \partial_n \left[x_1^{2\nu_1+1} x_2^{2\nu_2+1} \cdots x_n^{2\nu_n+1}\, \partial_n\left(x_1^{-\nu_1} x_2^{-\nu_2} \cdots x_n^{-\nu_n} J\right)\right] + \\ + x_1^{\nu_1+1} x_2^{\nu_2+1} \cdots x_n^{\nu_n+1} J = 0\,.$$

Further, we have

$$(1.9)\qquad \partial_n\left(x_1^{-\nu_1} x_2^{-\nu_2} \cdots x_n^{-\nu_n} J_{\nu_1,\nu_2,\cdots,\nu_n}\right)\Big|_{x_k=0} = 0 \qquad (k = 1, 2, \cdots, n),$$

by virtue of the conditions (1.6) and the second equation in (1.5).

In the following we shall consider the case $n = 2$ for the sake of simplicity. The passage to the general case is immediate.

2. Solution in the case $n = 2$

Consider the functional equations

$$(2.1)\qquad \frac{\partial^2}{\partial x\,\partial y}(x^{\mu} y^{\nu} J_{\mu,\nu}) = x^{\mu} y^{\nu} J_{\mu-1,\nu-1}\,,\quad \frac{\partial^2}{\partial x\,\partial y}(x^{-\mu} y^{-\nu} J_{\mu,\nu}) = -x^{-\mu} y^{-\nu} J_{\mu+1,\nu+1}$$

and the associated Mangeron equation

$$(2.2)\qquad \mathrm{L} J_{\mu,\nu} \equiv \frac{\partial^2}{\partial x\,\partial y}\left[x^{2\mu+1} y^{2\nu+1} \frac{\partial^2}{\partial x\,\partial y}(x^{-\mu} y^{-\nu} J_{\mu,\nu})\right] + x^{\mu+1} y^{\nu+1} J_{\mu,\nu} = 0\,.$$

We now establish the analytic solution to the above equations which satisfies the conditions

$$(2.3)\qquad (x^{-\mu} y^{-\nu} J_{\mu,\nu})\Big|_{x=0} = (x^{-\mu} y^{-\nu} J_{\mu,\nu})\Big|_{y=0} = \frac{1}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)}$$

and

$$(2.4)\qquad \frac{\partial^2}{\partial x\,\partial y}(x^{-\mu} y^{-\nu} J_{\mu,\nu})\Big|_{x=0} = \frac{\partial^2}{\partial x\,\partial y}(x^{-\mu} y^{-\nu} J_{\mu,\nu})\Big|_{y=0} = 0\,.$$

For this purpose, we shall seek a solution of the form

$$(2.5)\qquad J_{\mu,\nu}(x\,,y) = x^{\rho} y^{\sigma} \sum_{m,n=0}^{\infty} a_{m,n}\, x^m y^n \qquad (a_{0,0} \neq 0)$$

to the problem (2.2)–(2.4), where the indices $\rho = \rho(\mu\,,\nu)$ and $\sigma = \sigma(\mu\,,\nu)$, and the coefficients $a_{m,n} = a_{m,n}(\mu\,,\nu)$ are to be determined.

We can easily verify that

$$(2.6)\qquad LJ_{\mu,\nu} \equiv x^{\rho+\mu-1}\, y^{\sigma+\nu-1} \sum_{m,n=0}^{\infty} [f_{m,n}(\rho\,,\sigma)\, a_{m,n} + a_{m-2,n-2}]\, x^m y^n$$

where

$$(2.7)\qquad a_{m,-2} = a_{m,-1} = a_{-1,n} = a_{-2,n} = 0 \qquad (m\,,n = 0\,,1\,,2\,,\cdots)$$

and

$$(2.8)\qquad f_{m,n}(\rho\,,\sigma) = [(m+\rho)^2 - \mu^2]\,[(n+\sigma)^2 - \nu^2]\,.$$

Thus, Eq. (2.2) is satisfied if and only if

$$(2.9)\qquad f_{0,0}(\rho\,,\sigma)\, a_{0,0} = 0 \quad,\quad f_{m,n}(\rho\,,\sigma)\, a_{m,n} + a_{m-2,n-2} = 0 \quad (m,n=0,1,2,\cdots).$$

Since $a_{0,0} \neq 0$ by hypothesis, and $f_{0,0}(\rho\,,\sigma) = (\rho^2 - \mu^2)(\sigma^2 - \nu^2)$, we have

$$(2.10)\qquad \rho = \mp\mu \quad,\quad \sigma = \mp\nu$$

by virtue of the first equation in (2.9).

Assuming $\mathrm{Re}\,\mu \geq 0$ and $\mathrm{Re}\,\nu \geq 0$, we take $\rho = \mu$ and $\sigma = \nu$ in (2.5). The other three cases can be discussed by the well known method of Frobenius. In this case we have

$$(2.11)\qquad J_{\mu,\nu}(x\,,y) = x^{\mu} y^{\nu} \sum_{m,n=0}^{\infty} a_{m,n}\, x^m y^n$$

and

$$(2.12)\qquad mn(m+2\mu)(n+2\nu)\, a_{m,n} + a_{m-2,n-2} = 0 \qquad (m\,,n = 0\,,1\,,2\,,\cdots).$$

Furthermore,

$$(2.13)\qquad a_{0,0} = \frac{1}{2^{\mu+\nu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\ ,\quad a_{m,0} = a_{0,n} = 0 \qquad (m\,,n = 0\,,1\,,2\,,\cdots)$$

by virtue of the conditions (2.3), and

$$(2.14)\qquad a_{m,1} = a_{1,n} = 0 \qquad (m\,,n = 1\,,2\,,3\,,\cdots)$$

by the conditions (2.4). Thus, all the coefficients $a_{m,n}$ are uniquely determined by the recurrence relation (2.12) and by the initial conditions (2.13)–(2.14). Solving the system (2.12)–(2.16) we find that $a_{m,n} = 0$ for all m and n except $m = n = 2j$, for which we have

$$(2.15)\qquad a_{2j,2j} = (-1)^j \frac{1}{2^{\mu+\nu+4j}(j!)^2\,\Gamma(\mu+j+1)\,\Gamma(\nu+j+1)} \qquad (j = 0,1,2,\cdots).$$

Hence the solution of the problem (2.2)–(2.4) is of the form

$$(2.16)\qquad J_{\mu,\nu}(x\,,y) = \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{x}{2}\right)^{\mu+2j}\left(\frac{y}{2}\right)^{\nu+2j}}{(j!)^2\,\Gamma(\mu+j+1)\,\Gamma(\nu+j+1)}\,.$$

We can verify without any difficulty that the function $J_{\mu,\nu}(x\,,y)$ given by the formula (2.16) satisfies the functional equations (2.1) and the conditions (2.3). Obviously $x^{-\mu}y^{-\nu}J_{\mu,\nu}(x\,,y)$ is an entire function. Hence, if $\mathrm{Re}\,\mu > 0$, $\mathrm{Re}\,\nu > 0$, then $J_{\mu,\nu}(x\,,y)$ is the desired generalized Bessel function.

By a similar analysis we can show that the functions $J_{-\mu,\nu}(x\,,y)$, $J_{\mu,-\nu}(x\,,y)$ and $J_{-\mu,-\nu}(x\,,y)$ satisfy also the functional equations (2.1) and the conditions (2.3), under certain conditions which are usually imposed in ordinary Bessel functions of the first kind.

3. General case

Note that the solution of the general problem (1.5)–(1.6) is given by the formula

$$(3.1)\quad J_{\nu_1,\nu_2,\cdots,\nu_n}(x_1,x_2,\cdots,x_n) = \sum_{j=0}^{\infty}(-1)^j \frac{\left(\frac{x_1}{2}\right)^{\nu_1+2j}\left(\frac{x_2}{2}\right)^{\nu_2+2j}\cdots\left(\frac{x_n}{2}\right)^{\nu_n+2j}}{(j!)^n\,\Gamma(\nu_1+j+1)\,\Gamma(\nu_2+j+1)\cdots\Gamma(\nu_n+j+1)}$$

if $\mathrm{Re}\,\nu_j > 0$, $j = 1\,,2\,,\cdots,n$, which can be established by the same method used in the preceding section.

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