Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On unimodular contractions on Banach spaces and Hilbert spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.4, p. 444–447.

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<http://www.bdim.eu/item?id=RLINA_1971_8_50_4_444_0>

Analisi funzionale. — *On unimodular contractions on Banach spaces and Hilbert spaces.* Nota di IOANA ISTRĂȚESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota sarà dimostrato che se T è una contrazione con $\sigma(T) \subseteq \{z, |z| = 1\}$ di uno spazio di Banach X in sè, i vettori proprii che corrispondono ai valori proprii distinti sono ortogonali; ma quando X è spazio di Hilbert, il teorema di Weyl è valido per T.

1. Let X be a Banach space and T be an operator such that

1) $\|T\| \leq 1$

2) $\sigma(T) \subset \{z, |z| = 1\}$.

Call an arbitrary operator T with the above properties 1 and 2 a unimodular contraction (u.c.).

Our aim in this note is to obtain some results on operators which are u.c. defined on Banach spaces and also on Hilbert spaces.

2. Let X be a Banach space and T be a u.c. operator. We use the idea of G. Lumer, that one can construct a semi-inner-product [,], i.e., a mapping from $X \times X$ into C such that

(i) $[x, x] = \|x\|^2$

(ii) $[ax + by, z] = a[x, z] + b[y, z]$

(iii) $|[x, y]|^2 \leq \|x\|^2 \|y\|^2$

and that Giles has shown that it is possible to choose a semi-inner-product such that

(iv) $[x, ay] = \bar{a}[x, y]$.

We assume that all semi-inner-products satisfy (iv).

We shall follow James in saying that x is orthogonal to y if $\|x\| \leq \leq \|x + ay\|$ for all scalar a.

Our first result represents a generalization of well-known result for unitary operator that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

THEOREM 1. *If* T *is u.c. then eigenvectors of* T *corresponding to distinct eigenvalues are orthogonal.*

(*) Nella seduta del 17 aprile 1971.

Proof. Let glim be a Banach limit on 1^∞ and [,] be a semi-inner-product on X. We define

$$[[x , y]] = \text{glim}\, \{[T^n x , T^n y]\}$$

which has many properties of the semi-inner-product, i.e., the properties (ii)–(iv). We remark that for each x for which $\|T^n x\| = \|x\|$ then

$$[[x , x]] = \|x\|^2$$

Let x_0 and y_0 such that $Tx_0 = \alpha x_0$, $Ty_0 = \beta y_0$ and thus

$$[[x , y]] = [[Tx , Ty]]$$

by the translation invariance of glim. We have from this

$$[[x_0 , y_0]] = [[Tx_0 , Ty_0]] = \alpha\bar{\beta}\, [[x_0 , y_0]]$$

which gives that $(\alpha \neq \beta)$ $[[x_0 , y_0]] = 0$. Since

$$\|x_0\|^2 = [[x_0 , y_0]] = [[x_0 + ay_0 , x_0]] \leq \|x_0\| \, \|x_0 + ay_0\|$$

which shows that x_0 and y_0 are orthogonal. The theorem is proved.

The following theorem gives a method for obtaining examples of unimodular contractions.

THEOREM 2. *Let* T , S *be two unimodular contractions on Banach spaces* X *and* Y *respectively. Then* $T \otimes S$ *on* $X \otimes Y$ *is a unimodular contraction.*

Proof. The fact that $T \otimes S$ is a contraction is clear. The fact that $\sigma(T \otimes S) \subset \{z , |z| = 1\}$ follows from a recent result of M. Schechter

$$\sigma(T \otimes S) = \{\lambda\mu \,;\, \lambda \in \sigma(T) , \mu \in \sigma(S)\}$$

and the theorem is proved.

Remark. The above theorem gives a method for obtaining an example of non-normal operator whose spectrum is the unit circle and consists of point spectrum only.

Indeed, let U be a unitary operator on a Hilbert space such that $\sigma(U) = \{z , |z| = 1\}$. It is known that every point in the spectrum is in approximate point spectrum. Let us consider U in the Hilbert space constructed by Berberian [2] and consider the unimodular contraction T constructed by Russo [6]. The operator $U \otimes T$ is a non-unitary unimodular contraction with $\{z , |z| = 1\}$ in the point spectrum. This example is, of course, connected with Prop. 1.2. of Russo.

3. Our next result refers to the Weyl spectrum. If $\mathfrak{L}(H)$ denotes the Banach algebra of all bounded operators on a Hilbert space H and I (H) is the ideal of all compact operators then the Weyl spectrum of T is the set

$$\omega(T) = \bigcap_{K \in I(H)} \sigma(T + K).$$

We say that Weyl's theorem holds for T if

$$\omega(T) = \sigma - \pi_{00}(T)$$

where $\pi_{00}(T)$ is the set of all isolated eigenvalues of finite multiplicity of T. Our result is the following:

THEOREM. *If* T *is a unimodular contraction then Weyl's theorem holds for* T.

Proof. For this we use a recent result of Baxley. We must verify the conditions C_1 and C_2 of that paper.

The condition C_1 says that, if $\{\lambda_n\}$ is an infinite sequence of distinct points in $\pi_{of}(T)$ (= the set of all eigenvalues of finite multiplicity) and $\{x_n\}$ is any sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge.

This condition is clear for our class of operators since every space

$$E_{\lambda_i} = \{x , Tx = \lambda_i x\}$$

is reducing and $|\lambda_i| = 1$ for each $i = 1, 2, \cdots$. The condition C_2 is as follows: for each $\lambda \in \pi_{00}(T)$, $T - \lambda I$ has closed range and index zero.

First we show that $T - \lambda I$ has closed range.

If $H_\lambda = \{x , Tx = \lambda x\}$ (which is reducing for T) we consider

$$H = H_\lambda \oplus H_\lambda^\perp$$

and let $\{x_n\}$ such that $\{Tx_n - \lambda x_n\}$ is convergent. We can show that there exists x_0 such that $\lim (Tx_n - \lambda x_n) = Tx_0 - \lambda x_0$.

Indeed, for each n,

$$x_n = x_n^\lambda + x_n^{\lambda,\perp}$$

and thus

$$Tx_n - \lambda x_n = (T - \lambda)\, x_n^\lambda \oplus (T - \lambda)\, x_n^{\lambda,\perp} = 0 \oplus Tx_n^{\lambda,\perp} - \lambda x_n^{\lambda,\perp}.$$

But on $H_\lambda^\perp$, $T - \lambda$ is invertible, so that $x_n^{\lambda,\perp}$ has a limit, $x_0^{\lambda,\perp}$. It is clear that

$$\lim (Tx_n - \lambda x_n) = Tx_0^{\lambda,\perp} - \lambda x_0^{\lambda,\perp}$$

and the assertion is proved.

Since $\sigma(T) = \{\lambda\} \cup \sigma(T|_{N(T-\lambda)^\perp})$ it follows that the index of $T - \lambda$ is 0, and thus C_2 is satisfied.

Remark. Perhaps the theorem is valid for unimodular numerical contractions on Hilbert spaces.

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