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A generalization of a boundedness tkeorem for the equation $\dddot{x} + \alpha\ddot{x} + \varphi_2(\dot{x}) + \varphi_3(x) = \psi(t, x, \dot{x}, \ddot{x})$

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Equazioni differenziali ordinarie. — *A generalization of a boundedness theorem for the equation* $\dddot{x} + \alpha \ddot{x} + \varphi_2(\dot{x}) + \varphi_3(x) = \psi(t, x, \dot{x}, \ddot{x})$. Nota di JAMES OKOYE CHUKUKA EZEILO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Per le equazioni considerate in questa Nota, nel caso che α sia costante, $\varphi_2, \varphi_3, \psi$ siano dipendenti dagli argomenti indicati nel titolo della Nota, sono state date alcune «generalizzate» condizioni di Hurwitz [1], atte ad assicurare la definitiva limitatezza delle soluzioni.

Il principale scopo di questa Nota é di estendere i risultati precedenti nel caso che φ_2 dipenda da x e $\dot{x}$ e il coefficiente α sia funzione di $x, \dot{x}, \ddot{x}$.

1. In the equation in the title, which will be referred to in the sequel as the equation (E), α is a constant and φ_2, φ_3 and ψ are continuous functions depending only on the arguments shown.

In a previous paper [1] it was shown that if ψ is bounded and $\varphi_3'(x)$ continuous for all x and if further α, φ_2 and φ_3 satisfy certain explicitly given generalized " Routh-Hurwitz conditions " then solutions of the equation (E) are all ultimately bounded. The present note which has been inspired by an investigation by Harrow [2] (particularly by the remark concerning the case $|p(t)|$ bounded on page 588 of [2]) is directed to the situation when the coefficient α in (E) is replaced by *bounded* functions φ_1. It turns out that, where this class of φ_1 is involved, the boundedness result is readily extendable to the much more general equations of the form:

$$\dddot{x} + \varphi_1(x, \dot{x}, \ddot{x})\ddot{x} + \varphi_2(x, \dot{x}) + \varphi_3(x) = \psi(t, x, \dot{x}, \ddot{x}) \tag{1.1}$$

in which φ_3 and ψ are as before but φ_1 and φ_2 can also depend on the extra variables indicated. Indeed assume here that $\varphi_1, \varphi_2, \varphi_3$ and ψ are continuous in their various arguments; also that $\varphi_3'(x)$, $\frac{\partial \varphi_1}{\partial x}(x, y, z)$, $\frac{\partial \varphi_1}{\partial z}(x, y, z)$ and $\frac{\partial \varphi_2}{\partial x}(x, y)$ exist and are continuous for all x, y and z. We have

THEOREM. – *Suppose that*

(i) *there are positive constants* δ_1, Δ_1 *such that* $\delta_1 \leq \varphi_1(x, y, z) \leq \Delta_1$, *for all* x, y *and* z,

(ii) *there are constants* $\delta_2 > 0$, $\Delta_2 > 0$ *such that*

$$\varphi_2(x, y)/y \geq \delta_2 \quad (|y| \geq \Delta_2) \tag{1.2}$$

uniformly in x,

(iii) *there is a constant* $\Delta_3 > 0$, *such that* $\varphi_3'(x) \leq \delta_3$ *for* $|x| \geq \Delta_3$ *where* δ_3 *is a constant such that*

$$\delta_1 \delta_2 > \delta_3 > 0, \tag{1.3}$$

(*) Nella seduta del 13 marzo 1971.

(iv) $\varphi_3(x)\,\mathrm{sgn}\,x - 2\eta_2\gamma(x) \to +\infty$ *as* $|x| \to \infty$ *where,*

(1.4) $\eta_2 = \max(1, \Delta_2)$ *and* $\gamma(x) = \max_{|y|\leq\eta_2} |\varphi_2(x, y)|$,

(v) $y\dfrac{\partial\varphi_1}{\partial x}(x, y, 0) \leq 0$, $\dfrac{\partial\varphi_2}{\partial x}(x, y) \leq 0$ *and* $y\dfrac{\partial\varphi_1}{\partial z}(x, y, z) \geq 0$ *for all* x, y *and* z,

(vi) $|\psi(t, x, y, z)| \leq A < \infty$ *for all* t, x, y *and* z.

Then there exists a constant $D_0 > 0$ *whose magnitude depends only on* A, φ_1, φ_2 *and* φ_3 *such that every solution* $x(t)$ *of* (1.4) *satisfies*

(1.5) $$|x(t)| \leq D_0 \quad , \quad |\dot{x}(t)| \leq D_0 \quad , \quad |\ddot{x}(t)| \leq D_0,$$

for all sufficiently large t.

We shall see in § 6 that the methods can actually be extended to cover the case where the function ψ in the theorem satisfies

$$|\psi(t, x, y, z)| \leq A + \varepsilon(y^2 + z^2)^{1/2}$$

for all t, x, y and z, where $A > 0$ and $\varepsilon > 0$ are constants, with ε sufficiently small.

2. The notation and procedure to be adopted here for the proof of the theorem will be exactly as in [1].

Thus we shall use D's for positive constants whose magnitudes depend on A, φ_1, φ_2 and φ_3, subject to the usual understanding that no two D's are ever the same unless numbered, while the D's: $D_1, D_2, D_3, \cdots$ with suffixes attached retain their identities throughout.

Thus also, coming to the actual verification of (1.5) itself, it will suffice (for the same reasons as in § 3 of [1]) merely to turn to the equivalent differential system derived from (1.1) by setting $y = \dot{x}$ and $z = \ddot{x}$ and to show that there is a continuous function $V(x, y, z)$, satisfying

(2.1) $$V(x, y, z) \to +\infty \quad \text{as } x^2 + y^2 + z^2 \to \infty,$$

such that the limit

(2.2) $$\dot{V}^* \equiv \limsup_{h\to+0} \frac{V(x(t+h), y(t+h), z(t+h)) - V(x(t), y(t), z(t))}{h}$$

exists, corresponding to any solution $(x(t), y(t), z(t))$ of the equivalent differential system of (1.1), and satisfies

(2.3) $$\dot{V}^* \leq -D_1 \quad \text{if } x^2(t) + y^2(t) + z^2(t) \geq D_2,$$

for some constants D_1, D_2.

3. *A function* V. Assume henceforth that all the conditions of the Theorem hold.

Let $C \equiv \max_{|x|\leq\Delta_3} |\varphi_3'(x)|$ and let $\delta > 0$ be a constant fixed, as is possible in view of (1.3), such that

(3.1) $$\delta_2\,\delta_3^{-1} > \delta > \delta_1^{-1}.$$

Also let $\chi_2(\xi, \eta)$ be the continuous function given by

$$\chi_2 = \begin{cases} \xi \operatorname{sgn} \eta , & \text{if } |\eta| \geq |\xi| , \\ \eta \operatorname{sgn} \xi , & \text{if } |\xi| \geq |\eta| , \end{cases} \tag{3.2}$$

and let $\chi_3(x)$ be the differentiable function given by

$$\chi_3 = \begin{cases} \operatorname{sgn} x , & \text{if } |x| \geq 2\Delta_3 , \\ \sin \pi x/(4\Delta_3) , & \text{if } |x| \leq 2\Delta_3 . \end{cases} \tag{3.3}$$

Now let $V = V(x, y, z)$ be the continuous function given by

$$V = V_1 + V_2 - V_3 \tag{3.4}$$

where

$$2V_1 = 2\int_0^x \varphi_3(\xi)\, d\xi + \delta\left(2\int_0^y \varphi_2(x, \eta)\, d\eta + z^2\right) + \tag{3.5}$$

$$+ 2\int_0^y \eta\, \varphi_1(x, \eta, 0)\, d\eta + 2\delta y\, \varphi_3(x) + 2yz ,$$

$$V_2 = \chi_2(z, x), \tag{3.6}$$

and

$$V_3 = D_3\, y \chi_3(x), \tag{3.7}$$

where $D_3 = 8\Delta_3\delta_2 C/(\pi\delta_3)$. The whole point in our proof of the Theorem is to show that this function V does in fact fulfill the provisions (2.1) and (2.3).

4. *Verification of* (2.1).

We shall require the following inequalities:

$$\int_0^y \eta\, \varphi_1(x, \eta, 0)\, d\eta \geq \delta_1 y^2 \quad \text{for all } x, y \tag{4.1}$$

$$\gamma(x) \leq D(|x| + 1) \quad \text{for all } x, \tag{4.2}$$

$$2\int_0^y \varphi_2(x, \eta)\, d\eta \geq \delta_2 y^2 - 2\eta_2 \gamma(x) - D \quad \text{for all } x, y. \tag{4.3}$$

The result (4.1) is immediate since $\varphi_1 \geq \delta_1$; and (4.2) follows on combining the fact that $\gamma(x) \leq D(|\varphi_3(x)| + 1)$ (from hypothesis (iv)) with the fact that $|\varphi_3(x)| \leq \delta_3 |x| + D$ (itself a consequence of hypothesis (iii) and the fact that $\varphi_3(x) \operatorname{sgn} x > 0$ for sufficiently large $|x|$).

To verify (4.3) we consider the sign of the function

$$\theta(x, y) \equiv 2\int_0^y \varphi_2(x, \eta)\, d\eta - \delta_2 y^2 + \delta_2 \eta_2^2 + 2\eta_2 \gamma(x)$$

separately in each of the cases: $|y| \leq \eta_2, y > \eta_2, y < -\eta_2$. In the first case $|\varphi_2| \leq \gamma(x)$ and so $\theta(x, y) \geq 0$. In the case $y > \eta_2$ write

$$\int_0^y \varphi_2(x, \eta)\, d\eta = \left(\int_0^{\eta_2} + \int_{\eta_2}^{y}\right) \varphi_2(x, \eta)\, d\eta$$
$$\equiv I_0 + I_1,$$

say, and note that $|I_0| \leq \eta_2 \gamma(x)$, and also, by (1.2), that $I_1 \geq \frac{1}{2}\delta_2(y^2 - \eta_2^2)$ so that, on combining, we shall have here that $\theta \geq 0$. Similarly $\theta \geq 0$ if $y < -\eta_2$. Hence $\theta \geq 0$ always and this proves (4.2).

Turning now to the actual verification of (2.1), observe first from (3.6) and (3.7) that

$$|V_2| \leq |x| \quad , \quad |V_3| \leq D_3 |y|;$$

and then also from (3.5), (4.1) and (4.3) that

$$2V_1 \geq 2\int_0^x \varphi_3(\xi)\, d\xi + \delta(\delta_2 y^2 - 2\eta_2 \gamma(x) - D) + \delta z^2 + \delta_1 y^2 + 2\delta y \varphi_3(x) + 2yz$$

$$= \delta(z + \delta^{-1} y)^2 + (\delta_1 - \delta^{-1}) y^2 + \delta_2^{-1}\delta(\delta_2 y + \varphi_3(x))^2 +$$

$$+ \delta_2^{-1}\left(2\delta_2 \int_0^x \varphi_3(\xi)\, d\xi - \delta \varphi_3^2(x)\right) - 2\delta\eta_2 \gamma(x) - D.$$

The term $2\delta_2 \int_0^x \varphi_3(\xi)\, d\xi - \delta\varphi_3^2(x) \equiv W_0(x)$ occurring here has already been estimated in [1; § 5] and the results there (see particularly (5.3) and (5.4) of [1]), show that there are constants D_4, D_5 such that

$$W_0(x) \geq 2D_4 \int_{\Delta_3 \operatorname{sgn} x}^{x} \varphi_3(\xi)\, d\xi - D_5 \quad (|x| \geq \Delta_3) \tag{4.4}$$

Thus, on gathering the various results for V_1, V_2 and V_3, we have that

$$2V \geq \delta(z + \delta^{-1} y)^2 + (\delta_1 - \delta^{-1}) y^2 + \delta^{-1}[W_0(x) - D|x|] - \\ - 2D_3|y| - D, \tag{4.5}$$

for all x, y and z, where we have taken advantage of (4.2) to replace the term $-2\delta\eta_2\gamma(x)$, occurring in the estimate of $2V_1$, by $-D(|x| + 1)$.

By (4.4), we have that, if $x \geq \Delta_3$

$$W_0(x) - Dx \geq 2D_4 \int_{\Delta_3}^{x} \{\varphi_3(\xi) - D\}\, d\xi - D.$$

The integral on the right hand side here tends to $+\infty$ as $|x| \to \infty$, since $\varphi_3(x) \operatorname{sgn} x \to +\infty$ as $|x| \to \infty$ (in view of hypothesis (iv) of the theorem); and thus

$$W_0(x) - Dx \to +\infty \quad \text{as } x \to \infty.$$

Analogously it can be shown that

$$W_0(x) + Dx \to +\infty \quad \text{as } x \to -\infty$$

Hence

$$W_0(x) - D|x| \to +\infty \quad \text{as } |x| \to \infty, \tag{4.6}$$

and since $\delta_1 - \delta^{-1} > 0$, by (3.2), the result (2.1) then follows at once from (4.5) and (4.6), for our V.

5. *Verification of* (2.3). For this part, in line with the remarks in the latter part of § 2, we turn to the differential system:

$$\dot{x} = y, \dot{y} = z, \dot{z} = -\varphi_1(x, y, z) z - \varphi_2(x, y) - \varphi_3(x) + \psi(t, x, y, z) \tag{5.1}$$

derived from (1.1) on setting $y = \dot{x}$ and $z = \ddot{x}$.

Let $(x, y, z) \equiv (x(t), y(t), z(t))$ be any solution of (5.1). As in [1] the limit $\dot{V}^*$ (see (2.2)), corresponding to (x, y, z), clearly exists. An elementary calculation from (5.1), (3.4), (3.5), (3.6) and (3.7) will, in fact, show that its value can be set out in the form:

$$\dot{V}^* = \begin{cases} -U_1 + U_2 + (\delta z + y)\psi + D_3 z\chi_3(x) + y \operatorname{sgn} z, & \text{if } |z| \geq |x| \\ -U_1 + U_2 - \varphi_3(x) \operatorname{sgn} x - (z\varphi_1 + \varphi_2) \operatorname{sgn} x + \\ \qquad + D_3 z\chi_3(x) + (\delta z + y + \operatorname{sgn} x)\psi, & \text{if } |x| \geq |z| \end{cases} \tag{5.2}$$

where

$$U_1 = y\varphi_2(x, y) + \{D_3 \chi_3'(x) - \delta \varphi_3'(x)\} y^2 + \{\delta\varphi_1(x, y, z) - 1\} z^2,$$

$$U_2 = yz[\varphi_1(x, y, 0) - \varphi_1(x, y, z)] + \delta y \int_0^y \frac{\partial \varphi_2}{\partial x}(x, \eta)\, d\eta +$$

$$+ y \int_0^y \eta \frac{\partial \varphi_1}{\partial x}(x, \eta, 0)\, d\eta.$$

Observe that each of the three components in U_2 is non-negative: the two integrals as a result of the first two conditions in hypothesis (v) of the theorem, and the remaining component because, by the mean value theorem,

$$yz\{\varphi_1(x, y, 0) - \varphi_1(x, y, z)\} = -yz^2 \frac{\partial \varphi_1}{\partial z}(x, y, \tau z) \qquad (\text{for some } \tau \text{ in } 0,1),$$

$$\leq 0,$$

by the last condition in the same hypothesis (v).

Observe next that the coefficient of z^2 in W_1 satisfies $\delta\varphi_1 - 1 \geq \delta\delta_1 - 1 > 0$ by (3.1). From this and the non-negativeness of U_2 it is clear that, if we set

$$y\,\varphi_2(x, y) + \{D_3\,\chi_3'(x) - \delta\,\varphi_3'(x)\}\,y^2 \equiv U_3(x, y)$$

then

$$\text{(5.3)}\qquad \dot{V}^* \leq \begin{cases} -U_3(x, y) - D_6\,z^2 + D(|y| + |z|)\,, & \text{if } |z| \geq |x| \\ -U_3(x, y) - D_6\,z^2 - \varphi_3(x)\operatorname{sgn} x + |\varphi_2(x, y)| + \\ \qquad + D(|y| + |z| + 1)\,, & \text{if } |z| \leq |x| \end{cases}$$

where we have now used the results $|\chi_3| \leq 1$, $|\psi| \leq A$ and have also used the fact of the boundedness of φ_1 to majorize the term $-z\varphi_1 \operatorname{sgn} x$ by $D|z|$.

From the definitions of χ_3 and D_3 in § 3 it is readily checked that

$$\text{(5.4)}\qquad \{\delta\varphi_3'(x) - D_3\,\chi_3'(x)\}\,y^2 \leq \delta\delta_3\,y^2$$

for all x, y. There is equally no difficulty in verifying from the definitions of η and $\gamma(x)$ in (1.4) that

$$\text{(5.5)}\qquad -y\varphi_2(x, y) + |\varphi_2(x, y)| \leq -\delta_2(y^2 - |y|) + \delta_2\,\eta_2^2 + 2\,\eta_2\,\gamma(x),$$

$$\text{(5.6)}\qquad -y\,\varphi_2(x, y) \leq -\delta_2\,y^2 + \delta_2\,\eta_2^2 + \eta_2\,\gamma(x).$$

In the case of (5.5), for example, consider the function

$$W_1 \equiv -y\varphi_2(x, y) + |\varphi_2(x, y)| + \delta_2(y^2 - |y|) - \delta_2\,\eta_2^2 - 2\,\eta_2\,\gamma(x).$$

If $|y| \leq \eta_2$ then clearly

$$W_1 \leq -(\eta_2 - 1)\,\gamma(x) - (\eta_2^2 - y^2)\,\delta_2 - \delta_2\,|y| < 0;$$

while if $|y| > \eta_2$ then by hypothesis (ii) of the theorem, $y\,\varphi_2 = |y| \cdot |\varphi_2|$ and thus

$$\begin{aligned} -y\varphi_2 + |\varphi_2| &= -(|y| - 1)\,|\varphi_2| \\ &\leq -\delta_2(|y| - 1)\,|y|, \end{aligned}$$

from which it follows that

$$W_1 \leq -\delta_2\,\eta_2^2 - 2\,\eta_2\,\gamma(x) < 0.$$

Hence $W_1 < 0$ always and this proves (5.5). The inequality (5.6) can be verified analogously by considering the function

$$W_2 \equiv -y\varphi_2(x, y) + \delta_2\,y^2 - \delta_2\,\eta_2^2 - \eta_2\,\gamma(x).$$

From (5.4) and (5.6) it follows that

$$-U_3(x, y) \leq -D_7\,y^2 + \eta_2\,\gamma(x) + D,$$

and from (5.4) and (5.5) that

$$-U_3(x, y) + |\varphi_2(x, y)| \leq -D_7\,y^2 + 2\,\eta_2\,\gamma(x) + D(|y| + 1)$$

where $D_7 \equiv \delta_2 - \delta\delta_3 > 0$, by (3.1). Hence, by (5.3),

$$\dot{V}^* \leq \begin{cases} -(D_7 y^2 + D_6 z^2) + \eta_2 \gamma(x) + D(|y| + |z| + 1), & \text{if } |z| \geq |x|, \\ -(D_7 y^2 + D_6 z^2) - \varphi_3(x) \operatorname{sgn} x + 2\eta_2 \gamma(x) + \\ \qquad + D(|y| + |z| + 1) & \text{if } |z| \leq |x|. \end{cases}$$

But, by (4.2), if $|x| \leq |z|$ then $\gamma(x) \leq D(|z| + 1)$; and so our latest inequality for $\dot{V}^*$ may also be set out in the form:

$$\text{(5.7)} \qquad \dot{V}^* \leq \begin{cases} -(D_7 y^2 + D_6 z^2) + D(|y| + |z| + 1), & \text{if } |z| \geq |x| \\ -(D_7 y^2 + D_6 z^2) - \{\varphi_3(x) \operatorname{sgn} x - 2\eta_2 \gamma(x)\} + \\ \qquad + D(|y| + |z| + 1), & \text{if } |z| \leq |x|. \end{cases}$$

At this stage it is useful to recall the hypothesis (iv) which implies, among other things, that there is a D such that

$$\text{(5.8)} \qquad -\{\varphi_3(x) \operatorname{sgn} x - 2\eta_2 \gamma(x)\} \leq D,$$

for all x. From (5.7) and (5.8) it is clear that whichever of the two estimates in (5.7) is applicable to $\dot{V}^*$, a constant D_8 exists such that

$$\text{(5.9)} \qquad \dot{V}^* \leq -1 \quad \text{if } y^2 + z^2 > D_8^2.$$

Suppose now, however, that

$$\text{(5.10)} \qquad y^2 + z^2 \leq D_8^2$$

Under such circumstances $|x| > D_8$ would imply that $|x| > |z|$ in which case the lower estimate for $\dot{V}^*$ in (5.7) is applicable. In other words, if (5.10) holds, then so long as $|x| > D_8$, we have that

$$\dot{V}^* \leq -\{\varphi_3(x) \operatorname{sgn} x - 2\eta_2 \gamma(x)\} + D.$$

Since $\varphi_3(x) \operatorname{sgn} x - 2\eta_2 \gamma(x) \to +\infty$, as $|x| \to \infty$, it is clear from this inequality that there is a constant $D_9 > D_8$ such that

$$\text{(5.11)} \qquad \dot{V}^* \leq -1, \quad \text{if } y^2 + z^2 \leq D_8^2 \quad \text{but } |x| \geq D_9.$$

The results (5.9) and (5.11) show that

$$\dot{V}^* \leq -1, \quad \text{if } x^2 + y^2 + z^2 \geq D_8^2 + D_9^2$$

which thus verifies (2.3) and also concludes our proof of the Theorem.

6. *Further remarks*: There is no difficulty in extending the present methods to an equation (1.1) in which ψ satisfies

$$\text{(6.1)} \qquad |\psi(t, x, y, z)| \leq A + \varepsilon(y^2 + z^2)^{1/2},$$

with $A > 0$ and $\varepsilon > 0$ constants and ε sufficiently small. Indeed the replacement of hypothesis (vi) of the theorem by (6.1) does not affect the proof until the two estimates in (5.3) for $\dot{V}^*$, the right hand side of each part of which will now have to be augmented by a term of magnitude not exceeding $\varepsilon D (y^2 + z^2)$. However, since the dominant term involving y and z on the right hand side of each of the two estimates in (5.3) can be majorized by an expression of the form

$$-D(y^2 + z^2),$$

it is clear that, if ε is chosen small enough, the presence of these additional terms in $\dot{V}^*$, under the new hypothesis, will not affect the sign of each of the dominant terms, in question, so that the rest of the estimates for $\dot{V}^*$ can be validated once again under the new condition (6.1).

The absence of a term of the form $-Dx^2$ at any stage in our estimates for $\dot{V}^*$ has been responsible for the difficulty in extending results to the case where ψ satisfies the more general condition:

$$|\psi(t, x, y, z)| \leq A + \varepsilon (x^2 + y^2 + z^2)^{1/2}.$$

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[2] M. HARROW, « J. London Math. Soc. », *43* (1968), 587–592.