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On the solution of a non—linear mixed problem for the Navier-Stokes equations in a time dependent domain. Nota II

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Analisi matematica. — *On the solution of a non-linear mixed problem for the Navier-Stokes equations in a time dependent domain.* Nota II di GIOVANNI PROUSE, presentata (*) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dà la dimostrazione del Teorema 1 enunciato nella Nota I.

2. – In the present proof, as in the following ones, we shall often use some inequalities, which we recall below.

i) From the interpolation property for the spaces V_σ expressed by (1.7) it follows that, if $\alpha < \sigma < \beta$

$$(2.1) \qquad \|\vec{v}\|_{V_\sigma} \leq c_1 \|\vec{v}\|_{V_\alpha}^{\frac{\beta-\sigma}{\beta-\alpha}} \|\vec{v}\|_{V_\beta}^{\frac{\sigma-\alpha}{\beta-\alpha}} .$$

Moreover, the embedding of V_β in V_α, with $\beta > \alpha$ is completely continuous (Lions and Peetre [4]) (1).

ii) $\forall q , \varepsilon > 0$ it is

$$(2.2) \qquad |ab| \leq \varepsilon |a|^{1+q} + \varepsilon^{-1/q} |b|^{1+(1/q)}.$$

iii) Let B_1, B_2, B_3 be three Banach spaces, with $B_1 \subset B_2 \subset B_3$, the embedding of B_1 in B_2 being completely continuous. Then, $\forall \varepsilon > 0$, there exists a quantity $\delta(\varepsilon)$ such that (Lions [5])

$$(2.3) \qquad \|\vec{v}\|_{E_2} \leq \varepsilon \|\vec{v}\|_{E_1} + \delta(\varepsilon) \|\vec{v}\|_{E_3} .$$

We also wish to recall the following embedding (Nikolskii [6]) and trace (Prodi [7]) theorems.

iv) Let Ω be an open, bounded set of $\mathbf{R}^n$ satisfying the cone property; then:

$$(2.4) \qquad \begin{aligned} &\text{if } \frac{1}{2} - \frac{\alpha}{n} > 0 \qquad H^\alpha(\Omega) \subset L^q(\Omega) \qquad \text{with } \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{n} ; \\ &\text{if } \frac{1}{2} - \frac{\alpha}{n} < 0 \qquad H^\alpha(\Omega) \subset L^\infty(\Omega) . \end{aligned}$$

v) Let Γ be the boundary of Ω and γ denote the operator "trace on Γ" of a function $\vec{v}$ defined on Ω; then, if $\vec{v} \in H^\alpha(\Omega)$, with $\alpha > \frac{1}{2}$,

$$(2.5) \qquad \|\gamma \vec{v}\|_{L^q(\Gamma)} \leq c_2 \|\vec{v}\|_{H^\alpha(\Omega)} , \qquad \text{with } q = \frac{2(n-1)}{n-2\alpha} .$$

(*) Nella seduta del 20 febbraio 1971.

(1) The indications refer to the bibliography at the end of Note I.

Observe, finally, that it is obviously $V_1(\Omega) \subset H^1(\Omega)$; hence, being $V_0(\Omega)$ isomorphic to $L^2(\Omega)$, we obtain, by interpolation,

$$V_\sigma(\Omega) \subset H^\sigma(\Omega) \qquad \text{when} \quad 0 \leq \sigma \leq 1.$$

The embedding and trace theorems hold therefore also for the spaces $V_\sigma(\Omega)$, provided $0 \leq \sigma \leq 1$.

Let us now prove Theorem 1. We shall, for this, transform equation (1.10) in the following way.

Setting $\vec{u}(t) = e^{kt}\,\vec{v}(t)$, with $k > 0$, we obtain from (1.10)

$$\int_0^T \{ e^{kt} \langle \vec{v}'(t), \vec{h}(t)\rangle + \mu e^{kt} \langle A\vec{v}(t), \vec{h}(t)\rangle + $$
$$+ k e^{kt} \langle \vec{v}(t), \vec{h}(t)\rangle - \langle \vec{g}(t), \vec{h}(t)\rangle \}\, dt = 0 .$$

Hence, if $\vec{q}(t) = e^{-kt}\vec{g}(t)$, the function $\vec{v}(t)$ satisfies, $\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$, the equation

$$\int_0^T \{\langle \vec{v}'(t), \vec{h}(t)\rangle + \mu \langle A\vec{v}(t), \vec{h}(t)\rangle + \tag{2.6}$$
$$+ k(\vec{v}(t), \vec{h}(t))_{V_0(\Omega_t)} - \langle \vec{q}(t), \vec{h}(t)\rangle\}\, dt = 0$$

and, by (1.9), the initial condition

$$\vec{v}(0) = \vec{u}_0 . \tag{2.7}$$

In what follows we shall therefore substitute (1.10), (1.9) with (2.6), (2.7).

We begin by proving that there exists a function $\vec{v}(t) \in L^2(0, T; V_{\sigma+1}(\Omega_t))$ which satisfies the equation

$$\int_0^T \{-\langle \vec{v}(t), \vec{h}'(t)\rangle + \mu \langle A\vec{v}(t), \vec{h}(t)\rangle + \tag{2.8}$$
$$+ k(\vec{v}(t), \vec{h}(t))_{V_0(\Omega_t)} - \langle \vec{q}(t), \vec{h}(t)\rangle\}\, dt =$$
$$= \langle \vec{u}_0, \vec{h}(0)\rangle - \int_0^T \int_{\Gamma_{3,t}} \cos \vec{n}t \sum_{j=1}^{2} v_j(x, t)\, h_j(x, t)\, d\Gamma_{3,t}\, dt$$

($\vec{n}$ exterior normal to $\Gamma_{3,t} \times [0, T]$)

$\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$, with $\vec{h}'(t) \in L^2(0, T; V_{-\sigma}(\Omega_t))$, $\vec{h}(T) = 0$.

Observe that (2.8) is obtained from (2.6) integrating by parts with respect to the variable t.

Let us set, for simplicity, $F = L^2(0, T; V_{\sigma+1}(\Omega_t))$ and denote by Φ the space of functions $\vec{\varphi}(x, t)$ such that $\vec{\varphi} \in F$, $\vec{\varphi}'(t) \in L^2(0, T; V_\sigma(\Omega_t))$, $\vec{\varphi}(T) = 0$; in Φ we define a seminorm by the relation

$$\|\vec{\varphi}\|_\Phi^2 = \|\vec{\varphi}\|_F^2 + \|\vec{\varphi}(0)\|_{V_\sigma(\Omega_0)}^2 . \tag{2.9}$$

Moreover, we set

$$\begin{aligned} E(\vec{v}, \vec{\varphi}) = \int_0^T \Big\{ &-\langle \vec{v}(t), A^\sigma \vec{\varphi}'(t)\rangle + \mu \langle A\vec{v}(t), A^\sigma \vec{\varphi}(t)\rangle + \\ &+ k(\vec{v}(t), A^\sigma \vec{\varphi}(t))_{V_0(\Omega_t)} + \int_{\Gamma_{3,t}} \cos \vec{n t} \sum_{j=1}^2 v_j(x, t) A^\sigma \varphi_j(x, t)\, d\Gamma_{3,t} \Big\}\, dt \end{aligned} \tag{2.10}$$

$$L(\vec{\varphi}) = \int_0^T \langle \vec{q}(t), A^\sigma \vec{\varphi}(t)\rangle\, dt + \langle \vec{u}_0, A^\sigma \vec{\varphi}(0)\rangle .$$

It is then also

$$\begin{aligned} E(\vec{v}, \vec{\varphi}) = \int_0^T \Big\{ &-\langle \vec{v}(t), A^\sigma \vec{\varphi}'(t)\rangle + \mu \langle A\vec{v}(t), A^\sigma \vec{\varphi}(t)\rangle + \\ &+ k(\vec{v}(t), \vec{\varphi}(t))_{V_\sigma(\Omega_t)} + \int_{\Gamma_{3,t}} \cos \vec{n t} \sum_{j=1}^2 v_j(x, t) A^\sigma \varphi_j(x, t)\, d\Gamma_{3,t} \Big\}\, dt \end{aligned} \tag{2.11}$$

$$L(\vec{\varphi}) = \int_0^T \langle \vec{q}(t), A^\sigma \vec{\varphi}(t)\rangle\, dt + (\vec{u}_0, \vec{\varphi}(0))_{V_\sigma(\Omega_0)} .$$

By the embedding and trace theorems recalled above, it is evident that, $\forall \vec{\varphi}$ fixed in Φ, the linear form $\vec{v} \to E(\vec{v}, \vec{\varphi})$ is continuous on F; moreover, the linear form $\vec{\varphi} \to L(\vec{\varphi})$ is continuous on Φ. We have then

$$\begin{aligned} E(\vec{\varphi}, \vec{\varphi}) = \int_0^T \Big\{ &-(\vec{\varphi}(t), \vec{\varphi}'(t))_{V_\sigma(\Omega_t)} + \mu \|\vec{\varphi}(t)\|_{V_{\sigma+1}(\Omega_t)}^2 + \\ &+ k \|\vec{\varphi}(t)\|_{V_\sigma(\Omega_t)}^2 + \int_{\Gamma_{3,t}} \cos \vec{n t} \sum_{j=1}^2 \varphi_j(x, t) A^\sigma \varphi_j(x, t)\, d\Gamma_{3,t} \Big\}\, dt \end{aligned} \tag{2.12}$$

and, bearing in mind that $\vec{\varphi}(T) = 0$, it is

$$(2.13) \qquad \int_0^T (\vec{\varphi}(t), \vec{\varphi}'(t))_{V_\sigma(\Omega_t)} \, dt = \int_0^T \int_{\Omega_t} \sum_{j=1}^{2} A^{\sigma/2} \varphi_j(x, t) \frac{\partial A^{\sigma/2} \varphi_j(x, t)}{\partial t} \, d\Omega_t \, dt =$$

$$= \int_0^T \frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega_t} \sum_{j=1}^{2} (A^{\sigma/2} \varphi_j(x, t))^2 \, d\Omega_t \, dt -$$

$$- \int_0^T \int_0^l \frac{1}{2} \sum_{j=1}^{2} \Big[(A^{\sigma/2} \varphi_j(x_1, \psi_2(x_1, t), t))^2 \frac{\partial \psi_2}{\partial t} -$$

$$- (A^{\sigma/2} \varphi_j(x_1, \psi_1(x_1, t), t))^2 \frac{\partial \psi_1}{\partial t} \Big] \, dx_1 \, dt =$$

$$= -\frac{1}{2} \| \vec{\varphi}(0) \|^2_{V_\sigma(\Omega_0)} - \frac{1}{2} \int_0^T \int_0^l \sum_{j=1}^{2} \Big[(A^{\sigma/2} \varphi_j(x_1, \psi_2(x_1, t), t))^2 \frac{\partial \psi_2}{\partial t} -$$

$$- (A^{\sigma/2} \varphi_j(x_1, \psi_1(x_1, t), t))^2 \frac{\partial \psi_1}{\partial t} \Big] \, dx_1 \, dt .$$

Since $\frac{\partial \psi_i}{\partial t}$ are bounded, it follows from (2.13) and (2.5) that

$$(2.14) \qquad - \int_0^T (\vec{\varphi}(t), \vec{\varphi}'(t))_{V_\sigma(\Omega_t)} \, dt \geq$$

$$\geq \frac{1}{2} \| \vec{\varphi}(0) \|^2_{V_\sigma(\Omega_0)} - c_3 \int_0^T \| \gamma A^{\sigma/2} \vec{\varphi}(t) \|^2_{L^2(\Gamma_{3,t})} \, dt \geq$$

$$\geq \frac{1}{2} \| \vec{\varphi}(0) \|^2_{V_\sigma(\Omega_0)} - c_4 \int_0^T \| A^{\sigma/2} \vec{\varphi}(t) \|^2_{V_{(1/2)+\varepsilon}(\Omega_t)} \, dt \geq$$

$$\geq \frac{1}{2} \| \vec{\varphi}(0) \|^2_{V_\sigma(\Omega_0)} - c_5 \int_0^T \| \vec{\varphi}(t) \|^2_{V_1(\Omega_t)} ,$$

with $0 < \varepsilon \leq \frac{1}{2} - \sigma$.

Analogously,

$$(2.15) \qquad \left| \int_0^T \cos \vec{n} t \sum_{j=1}^{2} \varphi_j(x, t) \, A^\sigma \varphi_j(x, t) \, d\Gamma_{3,t} \, dt \right| \leq$$

$$\leq \int_0^T \|\gamma \vec{\varphi}(t)\|_{L^2(\Gamma_{3,t})} \|\gamma A^\sigma \vec{\varphi}(t)\|_{L^2(\Gamma_{3,t})} dt \leq$$

$$\leq c_6 \int_0^T \|\vec{\varphi}(t)\|_{V_{(1/2)+\varepsilon}(\Omega_t)} \|A^\sigma \vec{\varphi}(t)\|_{V_{(1/2)+\varepsilon}(\Omega_t)} dt \leq$$

$$\leq c_7 \int_0^T \|\vec{\varphi}(t)\|_{V_{(1/2)+\varepsilon}(\Omega_t)} \|\vec{\varphi}(t)\|_{V_{\sigma+1}(\Omega_t)} dt$$

since, if we choose $\varepsilon \leq \frac{1}{2} - \sigma$, it is $\frac{1}{2} + \varepsilon + 2\sigma \leq \sigma + 1$.

From (2.12), (2.14), (2.15) we obtain, by (2.1), (2.2), (2.3), for ε sufficiently small

$$(2.16)\quad E(\vec{\varphi}, \vec{\varphi}) \geq \int_0^T \{\mu \|\vec{\varphi}(t)\|^2_{V_{\sigma+1}(\Omega_t)} + k \|\vec{\varphi}(t)\|^2_{V_\sigma(\Omega_t)} - c_5 \|\vec{\varphi}(t)\|^2_{V_1(\Omega_t)} -$$

$$- c_7 \|\vec{\varphi}(t)\|_{V_{(1/2)+\varepsilon}(\Omega_t)} \|\vec{\varphi}(t)\|_{V_{\sigma+1}(\Omega_t)}\} dt + \frac{1}{2} \|\vec{\varphi}(0)\|^2_{V_\sigma(\Omega_0)} \geq$$

$$\geq \int_0^T \Big\{\mu \|\vec{\varphi}(t)\|^2_{V_{\sigma+1}(\Omega_t)} + k \|\vec{\varphi}(t)\|^2_{V_\sigma(\Omega_t)} - \frac{\mu}{4} \|\vec{\varphi}(t)\|^2_{V_{\sigma+1}(\Omega_t)} -$$

$$- c_8 \|\vec{\varphi}(t)\|^2_{V_\sigma(\Omega_t)} - \frac{\mu}{8} \|\vec{\varphi}(t)\|^2_{V_{\sigma+1}(\Omega_t)} - c_9 \|\vec{\varphi}(t)\|^2_{V_{(1/2)+\sigma}(\Omega_t)}\Big\} dt +$$

$$+ \frac{1}{2} \|\vec{\varphi}(0)\|^2_{V_\sigma(\Omega_0)} \geq \int_0^T \Big\{\frac{5}{8} \mu \|\vec{\varphi}(t)\|^2_{V_{\sigma+1}(\Omega_t)} + (k - c_8) \|\vec{\varphi}(t)\|^2_{V_\sigma(\Omega_t)} -$$

$$- \frac{1}{8} \mu \|\vec{\varphi}(t)\|^2_{V_{\sigma+1}(\Omega_t)} - c_{10} \|\vec{\varphi}(t)\|^2_{V_\sigma(\Omega_t)}\Big\} dt + \frac{1}{2} \|\vec{\varphi}(0)\|^2_{V_\sigma(\Omega_0)} .$$

If therefore we choose $k \geq c_8 + c_{10}$, it is

$$(2.17)\qquad E(\vec{\varphi}, \vec{\varphi}) \geq \min\left(\frac{\mu}{2}, \frac{1}{2}\right) \|\vec{\varphi}\|^2_\Phi .$$

By a well known theorem (see Lions [5] Ch. 3, Th. 1.1), there exists then, $\forall \vec{\varphi} \in \Phi$, a function $\vec{v} \in F$ such that

$$E(\vec{v}, \vec{\varphi}) = L(\vec{\varphi}),$$

i.e. such that

$$(2.18)\qquad \int_0^T \{-\langle \vec{v}(t), A^\sigma \vec{\varphi}'(t)\rangle + \mu \langle A\vec{v}(t), A^\sigma \vec{\varphi}(t)\rangle +$$

$$+ k(\vec{v}(t), A^\sigma \vec{\varphi}(t))_{V_0(\Omega_t)} - \langle \vec{q}(t), A^\sigma \vec{\varphi}(t)\rangle\} \, dt =$$

$$= \langle \vec{u}_0, A^\sigma \vec{\varphi}(0)\rangle - \int_0^T \int_{\Gamma_{3,t}} \cos \vec{n}t \sum_{j=1}^{2} v_j(x,t) A^\sigma \varphi_j(x,t) \, d\Gamma_{3,t} \, dt.$$

If we fix arbitrarily $\vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$, there exists, on the other hand, $\vec{\varphi}(t) \in L^2(0, T; V_{\sigma+1}(\Omega_t))$ such that

$$(2.19)\qquad \vec{h}(t) = A^\sigma \vec{\varphi}(t).$$

Hence, if we substitute (2.19) in (2.18), we obtain that $\vec{v}(t) \in F$ satisfies equation (2.8) $\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$ with $\vec{h}'(t) \in L^2(0, T; V_{-\sigma}(\Omega_t))$, $\vec{h}(T) = 0$.

Observe now that

$$\left| \int_0^T \{\mu \langle A\vec{v}(t), \vec{h}(t)\rangle + k(\vec{v}(t), \vec{h}(t))_{V_0(\Omega_t)} - \langle \vec{q}(t), \vec{h}(t)\rangle\} \, dt \right| \leq$$

$$\leq c_{11} (\|\vec{v}(t)\|_{L^2(0,T;V_{\sigma+1}(\Omega_t))} \|\vec{h}(t)\|_{L^2(0,T;V_{1-\sigma}(\Omega_t))} +$$

$$+ \|\vec{q}(t)\|_{L^2(0,T;V_{\sigma-1}(\Omega_t))} \|\vec{h}(t)\|_{L^2(0,T;V_{1-\sigma}(\Omega_t))})$$

and, consequently, there exists an operator $\Psi \in \mathfrak{L}(L^2(0, T; V_{1-\sigma}(\Omega_t)), L^2(0, T; V_{\sigma-1}(\Omega_t)))$ such that

$$(2.20)\qquad \int_0^T \{\mu \langle A\vec{v}(t), \vec{h}(t)\rangle + k(\vec{v}(t), \vec{h}(t))_{V_0(\Omega_t)} - \langle \vec{q}(t), \vec{h}(t)\rangle\} \, dt =$$

$$= \int_0^T \langle \Psi(t) \vec{v}(t), \vec{h}(t)\rangle \, dt.$$

Hence, by (2.8), (2.20),

$$\int_0^T -\langle \vec{v}(t), \vec{h}'(t)\rangle \, dt = \int_0^T \langle \Psi^{\cdot}(t)\vec{v}(t), \vec{h}(t)\rangle \, dt + \langle \vec{u}_0, \vec{h}(o)\rangle - \tag{2.21}$$

$$-\int_0^T \int_{\Gamma_{3,t}} \cos \vec{n}t \sum_{j=1}^{2} v_j(x,t)\, h_j(x,t)\, d\Gamma_{3,t}\, dt,$$

from which follows, by Green's formula, that

$$\vec{v}'(t) = \Psi^{\cdot}(t)\vec{v}(t) \in L^2(o, T; V_{\sigma-1}(\Omega_t)). \tag{2.22}$$

Equation (2.8) can then be written

$$\int_0^T \{\langle \vec{v}'(t), \vec{h}(t)\rangle + \mu \langle A\vec{v}(t), \vec{h}(t)\rangle + k(\vec{v}(t), \vec{h}(t))_{V_0(\Omega_t)} - \tag{2.23}$$

$$-\langle \vec{q}(t), \vec{h}(t)\rangle\} \, dt = o$$

where

$$\vec{v}(t) \in L^2(o, T; V_{\sigma+1}(\Omega_t)) \quad , \quad \vec{v}'(t) \in L^2(o, T; V_{\sigma-1}(\Omega_t)),$$

$$\vec{h}(t) \in L^2(o, T; V_{1-\sigma}(\Omega_t)) \quad , \quad \vec{h}'(t) \in L^2(o, T; V_{-\sigma}(\Omega_t)) \quad , \quad \vec{h}(T) = o;$$

since however the space of such functions is dense in $L^2(o, T; V_{1-\sigma}(\Omega_t))$, (2.23) will hold $\forall \vec{h}(t) \in L^2(o, T; V_{1-\sigma}(\Omega_t))$. By (2.21) it also follows that

$$\vec{v}(o) = \vec{u}_0 . \tag{2.24}$$

It remains to be shown that $\vec{v}(t) \in L^\infty(o, T; V_\sigma(\Omega_t))$.

Let us set $\vec{h}(t) = A^\sigma \vec{v}(t)$ when $t \leq \bar{t} \leq T$, $\vec{h}(t) = o$ when $\bar{t} < t$; this is obviously possible, since $A^\sigma \vec{v}(t) \in L^2(o, T; V_{1-\sigma}(\Omega_t))$. We obtain then from (2.23)

$$\int_0^{\bar{t}} \{\langle \vec{v}'(t), A^\sigma \vec{v}(t)\rangle + \mu \langle A\vec{v}(t), A^\sigma \vec{v}(t)\rangle + k(\vec{v}(t), A^\sigma \vec{v}(t))_{V_0(\Omega_t)} - \tag{2.25}$$

$$-\langle \vec{q}(t), A^\sigma \vec{v}(t)\rangle\} \, dt = o.$$

We have, on the other hand, analogously to (2.13),

$$(2.26)\qquad \int_0^{\bar t} \langle \vec{v}'(t), A^{\sigma} \vec{v}(t)\rangle \, dt = \frac{1}{2}\int_0^{\bar t} \frac{d}{dt} \|\vec{v}(t)\|^2_{V_{\sigma}(\Omega_t)} \, dt -$$

$$- \int_0^{\bar t}\int_0^{l} \sum_{j=1}^{2} \Big[(A^{\sigma/2} v_j(x_1, \psi_2(x_1, t), t))^2 \frac{\partial \psi_2}{\partial t} -$$

$$- (A^{\sigma/2} v_j(x_1, \psi_1(x_1, t), t))^2 \frac{\partial \psi_1}{\partial t} \Big] \, dx_1 \, dt .$$

Hence

$$(2.27)\qquad \frac{1}{2} \|\vec{v}(\bar t)\|^2_{V_{\sigma}(\Omega_{\bar t})} + \int_0^{\bar t} (\mu \|\vec{v}(t)\|^2_{V_{\sigma+1}(\Omega_t)} + k \|\vec{v}(t)\|^2_{V_{\sigma}(\Omega_t)}) \, dt \leq$$

$$\leq \frac{1}{2} \|\vec{u}_0\|^2_{V_{\sigma}(\Omega_0)} + \int_0^{\bar t} \{ \|\vec{q}(t)\|_{V_{\sigma-1}(\Omega_t)} \|\vec{v}(t)\|_{V_{\sigma+1}(\Omega_t)} + c_5 \|\vec{v}(t)\|^2_{V_1(\Omega_t)} \} \, dt ,$$

from which follows, applying the inequality $\|\vec{v}\|^2_{V_1} \leq \frac{\mu}{2} \|\vec{v}\|^2_{V_{\sigma+1}} + c_{12} \|\vec{v}\|^2_{V_{\sigma}}$, that $\vec{v}(t) \in L^{\infty}(0, T; V_{\sigma}(\Omega_t))$.

The theorem is therefore completely proved.