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## A remark on algebraic differential equations

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**Matematica.** — *A remark on algebraic differential equations.* Nota di KURT MAHLER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. – Sia $f$ una serie di potenze formale in un'indeterminata $z$, a coefficienti in un qualsiasi campo di caratteristica zero. Si dimostra che, se $f$ soddisfa ad un'equazione differenziale algebrica (alle derivate ordinarie rispetto alla $z$) e coefficienti in un campo di caratteristica zero, allora $f$ soddisfa di conseguenza ad un'equazione differenziale algebrica a coefficienti interi.

Consider an analytic function $w = f(z)$ which satisfies an algebraic differential equation

$$\mathrm{F}(z\,; w\,, w', \cdots, w^{(m)}) = 0\,.$$

Here F denotes a polynomial in $z\,, w\,, w', \cdots, w^{(m)}$ with coefficients that may depend on any set of parameters independent of $z$, and the order $m$ may be any non-negative integer.

It is clear that $w = f(z)$ satisfies not only $\mathrm{F} = 0$, but infinitely many other algebraic differential equations as well. We shall prove in this note that $w = f(z)$ in particular satisfies an algebraic differential equation

$$\mathrm{G}(z\,; w\,, w', \cdots, w^{(\mathrm{M})}) = 0$$

where G is a polynomial in $z\,, w\,, w', \cdots, w^{(\mathrm{M})}$ with *constant rational integral coefficients*, but of an order M which possibly may be greater than $m$. This is a rather surprising result because there exist only *countably many* distinct algebraic differential equations $\mathrm{G} = 0$ of this kind.

The proof is purely algebraic. It applies without change to formal power series with coefficients in any field of characteristic zero. Therefore only this more general case will be considered.

1. Let L be a field of characteristic zero, and $z$ an indeterminate. Denote by $\mathrm{L}^*$ the ring of all formal power series

$$f = \sum_{h=0}^{\infty} f_h z^h \quad , \quad g = \sum_{h=0}^{\infty} g_h z^h, \qquad \text{etc.},$$

in $z$ with coefficients $f_h\,, g_h$, etc., in L. Here the sum and the product of such series are as usual defined by

$$f + g = \sum_{h=0}^{\infty} (f_h + g_h) z^h \quad , \quad fg = \sum_{h=0}^{\infty} \left( \sum_{k=0}^{h} f_k\, g_{h-k} \right) z^h,$$

(*) Nella seduta del 17 aprile 1971.

and the elements $a$ of L are identified with the special power series

$$a = a + \sum_{h=1}^{\infty} 0 \cdot z^h$$

in $L^*$ and play the role of constants.

Differentiation in $L^*$ is defined formally by

$$\frac{d^k f}{dz^k} = f^{(k)} = \sum_{h=k}^{\infty} h(h-1)\cdots(h-k+1) f_h z^{h-k} \quad , \quad f^{(0)} = f.$$

It satisfies the usual rules for the derivatives of sums, differences, and products, and the equation

$$\frac{df}{dz} = 0$$

implies that $f = a$ is a constant, i.e. an element of L.

If

$$f = \sum_{h=0}^{\infty} f_h z^h$$

is any series in $L^*$, we denote by

$$K_f = Q(f_0, f_1, f_2, \cdots)$$

the field which is obtained by adjoining all the coefficients $f_h$ of $f$ to the rational number field $Q$. Thus $K_f$ is a subfield of L and depends on the particular power series $f$ which is studied.

To shorten the text, the term *Equation* with a capital E is always to mean "*algebraic differential equation*".

2. From now on let

$$f = \sum_{h=0}^{\infty} f_h z^h$$

be a fixed power series in $L^*$ which satisfies an Equation

$$\text{(F)} \qquad F(z; w, w', \cdots, w^{(m)}) = 0$$

of arbitrary order $m \geq 0$ and with coefficients in L. On differentiating this Equation repeatedly and applying algebraic operations to the results, we can obtain infinitely many other Equations for $f$ over L, i.e. with coefficients in L.

Whenever the polynomial F can be factorised into a product of polynomials with coefficients in L, at least one of the factors vanishes at $w = f$. If suffices therefore to consider only those Equations (F) in which F is an *irreducible* polynomial over L. We are, in addition, allowed to assume that (F) is of lowest possible order $m$, and that among all Equations for $f$ over L of this order $m$, it is also of lowest possible degree in $w^{(m)}$, the degree $n$ say.

The Equation for $f$ so defined is unique up to a factor in L. For if there were two such Equations, $F = 0$ and $F^* = 0$ say, we could eliminate the terms in $w^{(m)n}$ from them and obtain a new Equation $F^{**} = 0$ for $f$ over L which has either lower order $m$ or lower degree $n$ than $F = 0$.

The Equation (F) for $f$ over L fixed by these two properties of being irreducible and having smallest $m$ and $n$ will be called *the defining Equation for $f$ over* L.

To simplify the notation, we write from now on

$$F((w)) = F(z; w, w', \cdots, w^{(m)})$$

and put

$$F_j((w)) = \frac{\partial}{\partial w^{(j)}} F(z; w, w', \cdots, w^{(m)}) \qquad (j = 0, 1, \cdots, m),$$

and similarly for other differential polynomials. If (F) is the defining Equation for $f$ over L, then evidently

$$F((f)) = 0, \qquad \text{but} \quad F_m((f)) \neq 0.$$

For $F_m((w))$ does not vanish identically and has either lower order, or the same order but lower degree, than $F((w))$.

3. Let again (F) be the defining Equation for $f$ over L. In explicit form, $F((w))$ is a finite sum

$$F((w)) = \sum_{\varrho=1}^{r} F_\varrho z^{\nu_\varrho} w^{\nu_{\varrho 0}} w'^{\nu_{\varrho 1}} \cdots w^{(m)^{\nu_{\varrho m}}}$$

of monomials where the coefficients $F_\varrho$ lie in L and the $\nu$'s are non-negative integers. On substituting $w = f$, the monomials take the form

$$z^{\nu_\varrho} f^{\nu_{\varrho 0}} f'^{\nu_{\varrho 1}} \cdots f^{(m)^{\nu_{\varrho m}}} = \sum_{h=0}^{\infty} \Upsilon_{\varrho h} z^h \qquad (\rho = 1, 2, \cdots, r),$$

of power series in $K_f^*$, and the one Equation

$$F((f)) = 0$$

changes into the infinite system of homogeneous linear equations

$$\sum_{\varrho=1}^{r} F_\varrho \Upsilon_{\varrho h} = 0 \qquad (h = 0, 1, 2, \cdots)$$

for the coefficients $F_\varrho$ of $F((w))$. Since $F((w))$ is not identically zero, these equations possess a solution $F_1, \cdots, F_r$ distinct from the trivial solution $0, \cdots, 0$. On the other hand, the coefficients $\Upsilon_{\varrho h}$ of the linear equations lie in $K_f$. It follows then from linear algebra that there exists also a set of elements $F_1^*, \cdots, F_r^*$ of $K_f$ distinct from $0, \cdots, 0$ such that

$$\sum_{\varrho=1}^{r} F_\varrho^* \Upsilon_{\varrho h} = 0 \qquad (h = 0, 1, 2, \cdots).$$

Denote by $F^*$ the new differential polynomial

$$F^*((w)) = \sum_{\varrho=1}^{r} F_\varrho^* z^{\nu_\varrho} w^{\nu_{\varrho 0}} w'^{\nu_{\varrho 1}} \cdots w^{(m)^{\nu_{\varrho m}}}.$$

Then also $F^*$ is not identically zero, and $f$ satisfies the Equation

$$F^*((w)) = 0$$

over $K_f$. The reduction process of § 2, with L replaced by $K_f$, may now be applied to this Equation and leads to the result that

*$f$ has a defining Equation over $K_f$.*

From now on we shall assume that (F) itself is already this defining Equation over $K_f$.

4. On differentiating the formula $F((f)) = 0$ repeatedly, afterwards making use of the inequality $F_m((f)) \neq 0$ and putting $z = 0$, it can be proved that the coefficients $f_h$ of $f$ satisfy a recursive formula

$$A(h) f_h = \varphi_h (f_0, f_1, \cdots, f_{h-1}) \tag{2}$$

as soon as the suffix $h$ is sufficiently large. Here $A(h)$ is a polynomial in $h$ with coefficients in $K_f$ which does not vanish identically. Further, for each suffix $h$, $\varphi_h (f_0, f_1, \cdots, f_{h-1})$ is a polynomial in $f_0, f_1, \cdots, f_{h-1}$, with coefficients that are linear forms in the coefficients of $F((w))$ with rational coefficients, hence also lie in $K_f$. This result seems to be due to A. Hurwitz (1889). Detailed proofs can be found, e.g., in the Ph. D. thesis by Jan Popken (1935), or in my recent paper (Mahler 1971).

Denote by $h_0$ the smallest integer such that both

$$A(h) \neq 0 \qquad \text{for} \quad h \geq h_0,$$

and that the recursive formula (2) holds for $h \geq h_0$. This formula allows then to express successively all coefficients $f_h$ with $h \geq h_0$ rationally with rational coefficients in

(i) the finitely many coefficients of $F((w))$ and $A(h)$, and

(ii) the coefficients $f_0, f_1, \cdots, f_{h_0-1}$.

Since $K_f = Q(f_0, f_1, f_2, \cdots)$, we deduce then immediately the important consequence that

*The field $K_f$ is a finite extension of $Q$.*

This extension naturally may be algebraic or transcendental.

5. From general field theory, the relation between $Q$ and $K_f$ can be described in the following more explicit form.

*The field* $K_f$ *can be obtained as a finite extension*

$$K_f = Q(s_1, \cdots, s_r, t)$$

*of the rational number field* $Q$. *Here* $s_1, \cdots, s_r$ *are finitely many elements of* $K_f$ *which are algebraically independent over* $Q$, *and* $t$ *is a further element of* $K_f$ *which is algebraic over the intermediate extension field* $J = Q(s_1, \cdots, s_r)$.

The integer $r$ is the degree of transcendency of $K_f$ over $Q$ and is a non-negative integer. It may have the value $r = 0$, in which case $K_f$ is an algebraic number field of finite degree over $Q$.

The element $t$ of $K_f$ can always be chosen so as to be entire over the polynomial ring $P = Q[s_1, \cdots, s_r]$. The irreducible algebraic equation for $t$ over this ring has then the form

$$t^d + e_1(s_1, \cdots, s_r)\, t^{d-1} + \cdots + e_d(s_1, \cdots, s_r) = 0. \tag{3}$$

Here the degree $d$ is some positive integer, and the coefficients

$$e_1(s_1, \cdots, s_r), \cdots, e_d(s_1, \cdots, s_r)$$

are polynomials in the polynomial ring P. Let

$$t^{(0)} = t, t^{(1)}, \cdots, t^{(d-1)}$$

be the $d$ roots of this equation (3).

6. By hypothesis, $F((w))$ is irreducible over $K_f$ and has coefficients in $K_f$. Any non-zero factors of $F((w))$ not involving $z, w, w', \cdots, w^{(m)}$ are irrelevant and may be omitted. Therefore, without loss of generality, we can write

$$F((w)) = \Phi(z; w, w', \cdots, w^{(m)}; s_1, \cdots, s_r, t) = \Phi((w \mid s_1, \cdots, s_r, t)),$$

where $\Phi$ is a polynomial in $z, w, w', \cdots, w^{(m)}, s_1, \cdots, s_r$, and $t$, which is not identically zero and has rational coefficients.

Here we can remove $t$ by forming the norm

$$\Psi(z; w, w', \cdots, w^{(m)}; s_1, \cdots, s_r) = \Psi((w \mid s_1, \cdots, s_r)) = \prod_{\delta=0}^{d-1} \Phi((w \mid s_1, \cdots, s_r, t^{(\delta)})).$$

From the form of the equation (3) for $t^{(\delta)}$, $\Psi((w \mid s_1, \cdots, s_r))$ is then a polynomial in $z, w, w', \cdots, w^{(m)}, s_1, \cdots, s_r$ with rational coefficients which does not vanish identically, and $f$ satisfies the Equation

$$\Psi((w \mid s_1, \cdots, s_r)) = 0.$$

This is an Equation for $f$ over the field $J = Q(s_1, \cdots, s_r)$. The reduction process in § 2 enables us to derive from it also a defining Equation

$$X(z; w, w', \cdots, w^{(\mu)}; s_1, \cdots, s_r) = X((w \mid s_1, \cdots, s_r)) = 0 \tag{4}$$

for $f$ over J. Here X is an irreducible polynomial in $z, w, w', \cdots, w^{(\mu)}, s_1, \cdots, s_r$ with rational coefficients, of smallest order $\mu$ and smallest degree $\nu$ say, which vanishes for $w = f$.

7. The Equation (4) may still involve the $r$ quantities $s_1, \cdots. s_r$ in $K_f$ which, by hypothesis, are algebraically independent over $Q$; or one or more of these quantities may have disappeared in the process of forming the Equation (4). Let us assume that there exist a positive integer $\rho$ and $\rho$ suffixes $r_1, \cdots, r_\rho$ satisfying

$$1 \leq \rho \leq r \quad , \quad 1 \leq r_1 < r_2 < \cdots < r_\rho \leq r$$

such that there exists a defining Equation for $f$ over the field $Q(s_{r_1}, \cdots, s_{r_\rho})$, but that there is no such defining Equation over any subfield $Q(s_{r_1}, \cdots, s_{r_{\rho-1}})$ containing at most $\rho - 1$ of the quantities $s_{r_1}, \cdots, s_{r_\rho}$.

The defining Equation for $f$ over $Q(s_{r_1}, \cdots, s_{r_\rho})$ has the form

$$Y(z; w, w', \cdots, w^{(m)}; s_{r_1}, \cdots, s_{r_\rho}) = Y((w \mid s_{r_1}, \cdots, s_{r_\rho})) = 0,$$

where the differential polynomial Y, say of order $m$, is an irreducible polynomial in $z, w, w', \cdots, w^{(m)}, s_{r_1}, \cdots, s_{r_\rho}$ with rational coefficients. From the definition, Y contains the quantity $s_{r_\rho}$ explicitly,

$$\frac{\partial Y}{\partial s_{r_\rho}} \neq 0.$$

We form the further partial derivatives

$$Y_z((w \mid s_{r_1}, \cdots, s_{r_\rho})) = \frac{\partial}{\partial z} Y((w \mid s_{r_1}, \cdots, s_{r_\rho})),$$

$$Y_j((w \mid s_{r_1}, \cdots, s_{r_\rho})) = \frac{\partial}{\partial w^{(j)}} Y((w \mid s_{r_1}, \cdots, s_{r_\rho})) \qquad (j = 0, 1, \cdots, m),$$

and put

$$Y^*((w \mid s_{r_1}, \cdots, s_{r_\rho})) = Y_z((w \mid s_{r_1}, \cdots, s_{r_\rho})) + \sum_{j=0}^{m} Y_j((w \mid s_{r_1}, \cdots, s_{r_\rho}))\, w^{(j+1)}.$$

Then

$$Y^*((w \mid s_{r_1}, \cdots, s_{r_\rho})) = \frac{d}{dz} Y((w \mid s_{r_1}, \cdots, s_{r_\rho})) \qquad \text{identically in } w.$$

By hypothesis, $w = f$ satisfies the Equation

(5) $$Y((w \mid s_{r_1}, \cdots, s_{r_\rho})) = 0$$

and hence also the Equation

(6) $$Y^*((w \mid s_{r_1}, \cdots, s_{r_\rho})) = 0.$$

On the other hand, since (5) is a defining equation for $f$ and is of order $m$, necessarily

(7) $$Y_m((f \mid s_{r_1}, \cdots, s_{r_\rho})) \neq 0.$$

8. Next denote by

$$Z((w \mid s_{r_1}, \cdots, s_{r_{\varrho-1}}))$$

the resultant with respect to the quantity $s_{r_\varrho}$ of the two polynomials

$$(8) \qquad Y((w \mid s_{r_1}, \cdots, s_{r_\varrho})) \quad \text{and} \quad Y^*((w \mid s_{r_1}, \cdots, s_{r_\varrho})).$$

The two equations (5) and (6) for $f$ imply that

$$Z((f \mid s_{r_1}, \cdots, s_{r_{\varrho-1}})) = 0.$$

On the other hand, by hypothesis, $f$ does not possess a defining Equation over $Q(s_{r_1}, \cdots, s_{r_{\varrho-1}})$. The construction in § 2 shows therefore that $f$ does not satisfy any non-trivial Equation over this field. Hence it follows that

$$Z((w \mid s_{r_1}, \cdots, s_{r_{\varrho-1}})) = 0 \qquad \textit{identically in } w.$$

We deduce from this identity that the two polynomials (8) have a common factor which involves at least one of the symbols $z, w, w', \cdots, w^{(m)}$, $s_{r_1}, \cdots, s_{r_\varrho}$. Since Y is irreducible, this requires that $Y^*$ is divisible by Y. But Y does not involve the derivative $w^{(m+1)}$, while $Y^*$ depends on this derivative in the term

$$Y_m((w \mid s_{r_1}, \cdots, s_{r_\varrho}))\, w^{(m+1)}$$

which, by (7), does not vanish identically. It follows that also $Y_m$ is divisible by Y. However, $Y_m$ either is of lower order or of lower degree than Y, and so a contradiction arises.

This proves that the minimum assumption about $\varrho$ in § 7 is false, i.e. no such positive integer exists. Hence we arrive at the following result.

THEOREM: *Let*

$$f = \sum_{h=0}^{\infty} f_h z^h$$

*be a formal power series, with coefficients $f_h$ in any field of characteristic zero which satisfies some algebraic differential equation*

$$F(z; w, w', \cdots, w^{(m)}) = 0,$$

*again with coefficients in some field of characteristic zero. Then $f$ also satisfies an algebraic differential equation*

$$G(z; w, w', \cdots, w^{(M)}) = 0$$

*with constant rational integral coefficients.*

The set of all Equations $G = 0$ with rational coefficients evidently is countable. Thus, although the series $f$ certainly do not form a countable set, each such series satisfies one of a countable set of Equations

$$G_l(z; w, w', \cdots, w^{(M_l)}) = 0 \qquad (l = 1, 2, 3, \cdots).$$

This property holds in particular in the special case when the coefficients $f_h$ are complex numbers and $f$ has a circle of convergence, hence when $f = f(z)$ defines an analytic function of $z$.

9. We had found that

$$K_f = Q(f_0, f_1, f_2, \cdots) = Q(s_1, \cdots, s_r, t).$$

Here $s_1, \cdots, s_r$ are finitely many elements of $K_f$ which are algebraically independent over $Q$, and $t$ is a root of the irreducible equation

$$t^d + e_1(s_1, \cdots, s_r)\, t^{d-1} + \cdots + e_d(s_1, \cdots, s_r) = 0. \tag{3}$$

In terms of $s_1, \cdots, s_r$, and $t$, each coefficient $f_h$ of $f$ can be written in the form

$$f_h = \sum_{\delta=0}^{d-1} r_{h\delta}(s_1, \cdots, s_r)\, t^\delta, \qquad = R_h(s_1, \cdots, s_r, t) \quad \text{say}, \tag{9}$$

where the $r_{h\delta}$ are rational functions of $s_1, \cdots, s_r$ with rational coefficients.

Denote now by $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r$ a set of $r$ independent indeterminates over $Q$, and by $\boldsymbol{t}$ the algebraic function of $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r$ defined by the equation

$$\boldsymbol{t}^d + e_1(\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r)\, \boldsymbol{t}^{d-1} + \cdots + e_d(\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r) = 0. \tag{10}$$

Further put

$$\boldsymbol{f}_h = \sum_{\delta=0}^{d-1} r_{h\delta}(\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r)\, \boldsymbol{t}^\delta = R_h(\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r, \boldsymbol{t}), \tag{11}$$

and denote by $\boldsymbol{f}$ the formal power series

$$\boldsymbol{f} = \sum_{h=0}^{\infty} \boldsymbol{f}_h z^h.$$

This series is associated with the algebraic function field

$$K_{\boldsymbol{f}} = Q(\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{f}_2, \cdots) = Q(\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r, \boldsymbol{t})$$

which, for $r = 0$, becomes an algebraic number field.

By hypothesis, the Equation

$$G(z; w, w', \cdots, w^{(M)}) = 0$$

for the series $w = f$ has rational coefficients which are independent of $s_1, \cdots, s_r$, and $t$. It is also clear that the isomorphic mapping

$$(s_1, \cdots, s_r, t) \to (\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r, \boldsymbol{t})$$

preserves all rational relations over $Q$. It follows therefore that not only $w = f$, but also

$w = \boldsymbol{f}$ *satisfies the Equation* (G).

We say that $\boldsymbol{f}$ is an *indeterminate solution* of the Equation, to distinguish it from the *determinate solution* $f$ from which we started.

10. An advantageous way of selecting the indeterminates $s_1, \cdots, s_r$ of $f$, and correspondingly the indeterminates $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r$ of $\boldsymbol{f}$, is as follows.

Take $s_1 = f_{h_1}$ where $h_1 \geq 0$ is the smallest suffix such that $f_{h_1}$ is transcendental over $Q$. Next take $s_2 = f_{h_2}$ where $h_2 > h_1$ is the smallest suffix such that $f_{h_2}$ is transcendental over $Q(f_{h_1})$. Continuing in this manner, finally select a smallest suffix $h_r > h_{r-1}$ such that $f_{h_r}$ is transcendental over $Q(f_{h_1}, \cdots, f_{h_{r-1}})$, but that $f_h$ for $h > h_r$ is algebraic over $Q(f_{h_1}, \cdots, f_{h_r})$. This construction fixes the algebraically independent quantities

$$s_1 = f_{h_1}, \cdots, s_r = f_{h_r}, \qquad \text{where} \quad 0 \leq h_1 < h_2 < \cdots < h_r$$

of $K_f$, and $t$ in

$$K_f = Q(f_{h_1}, \cdots, f_{h_r}, t)$$

can then be chosen so as to satisfy an irreducible equation (3) with polynomial coefficients.

For the indeterminate solution $\boldsymbol{f}$ we find similarly that

$$\boldsymbol{s}_1 = \boldsymbol{f}_{h_1}, \cdots, \boldsymbol{s}_r = \boldsymbol{f}_{h_r} \qquad \text{and} \qquad K_{\boldsymbol{f}} = Q(\boldsymbol{f}_{h_1}, \cdots, \boldsymbol{f}_{h_r}, \boldsymbol{t}). \tag{12}$$

The ordered set

$$\{h\} = \{h_1, h_2, \cdots, h_r\}$$

is called the *suffix set* of both $f$ and $\boldsymbol{f}$, and $r$ is its dimension. Both $\{h\}$ and $r$ vary for the different solution of (G), and $\{h\}$ may have infinitely many distinct possibilities.

To give an example, the Equation

$$ww' w''' + w'^2 w'' - 2 ww''^2 = 0$$

has amongst others the special solutions

$$f = f_h z^h \qquad (h = 0, 1, 2, \cdots)$$

which belong to the suffix sets $\{0\}$, $\{1\}$, $\{2\}, \cdots$, respectively.

By making use of the indeterminate solutions and choosing the indeterminates $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r$ as in (12), it can be proved that

$$0 \leq r \leq M$$

for every Equation (G) of order M. This generalises the classical theorem of analysis that the general integral of an Equation of order M depends on M constants of integration.

11. Let as before

$$G(z; w, w', \cdots, w^{(M)}) = 0 \tag{G}$$

be the defining Equation for both $f$ and $\boldsymbol{f}$. The derivatives

$$\left(\frac{d}{dz}\right)^h G(z; w, w', \cdots, w^{(M)}), = G^{(h)}(z; w, w', \cdots, w^{(M+h)}) \text{ say,}$$

are polynomials with rational coefficients in $z, w, w', \cdots, w^{(M+h)}$, and $f$ and $\boldsymbol{f}$ satisfy the relations

$$G^{(h)}(z; f, f', \cdots, f^{(M+h)}) = 0 \quad \text{and} \quad G^{(h)}(z; \boldsymbol{f}, \boldsymbol{f}', \cdots, \boldsymbol{f}^{(M+h)}) = 0$$
$$(h = 0, 1, 2, \cdots).$$

Here substitute $z = 0$ so that $f^{(k)}$ and $\boldsymbol{f}^{(k)}$ become $k!f_k$ and $k!\boldsymbol{f}_k$, respectively. We obtain then the two infinite systems of algebraic equations

$$(13) \qquad G^{(h)}(0; f_0, 1!f_1, 2!f_2, \cdots, (M+h)!f_{M+h}) = 0 \qquad (h = 0, 1, 2, \cdots)$$

and

$$(14) \qquad G^{(h)}(0; \boldsymbol{f}_0, 1!\boldsymbol{f}_1, 2!\boldsymbol{f}_2, \cdots, (M+h)!\boldsymbol{f}_{M+h}) = 0 \qquad (h = 0, 1, 2, \cdots).$$

It is convenient to interpret these formulae geometrically by considering $f$ and $\boldsymbol{f}$ as points in an infinite dimensional space S, with the coordinates $f_0, f_1, f_2, \cdots$ and $\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{f}_2, \cdots$, respectively. By (13) and (14), these points lie on a manifold, **M** say.

This manifold is essentially finite dimensional and algebraic. For by means of similar considerations as in § 4 it can be proved that there exists a positive integer $k_0$, and that for every suffix $k \geq k_0$ there is a polynomial

$$\psi_k(w_0, w_1, \cdots, w_{k-1})$$

with rational coefficients, such that

$$(15) \qquad f_k = \psi_k(f_0, f_1, \cdots, f_{k-1}) \quad \text{and} \quad \boldsymbol{f}_k = \psi_k(\boldsymbol{f}_0, \boldsymbol{f}_1, \cdots, \boldsymbol{f}_{k-1}) \qquad \text{for } k \geq k_0.$$

Thus the first $k_0$ coordinates of the points $f$ and $\boldsymbol{f}$ determine all the others rationally, entirely, and with rational coefficients. However, the integer $r_0$ and the polynomials $\psi_k$ may depend on the particular solutions $f$ and $\boldsymbol{f}$.

12. One special type of Equation is of particular interest.

Let us assume that in the defining Equation

$$(G) \qquad G(w, w', \cdots, w^{(M)}) = 0$$

for $f$ and $\boldsymbol{f}$ the polynomial G does not involve $z$ explicitly and has rational integral coefficients. Also let this Equation have the exact order M so that the partial derivative

$$G_M(w, w', \cdots, w^{(M)}) = \frac{\partial}{\partial w^{(M)}} G(w, w', \cdots, w^{(M)})$$

does not vanish identically. Let us consider the first M coefficients $\boldsymbol{f}_0, \boldsymbol{f}_1, \cdots, \boldsymbol{f}_{M-1}$ of $\boldsymbol{f}$ as independent indeterminates. The algebraic equation

$$G(\boldsymbol{f}_0, 1!\boldsymbol{f}_1, \cdots, M!\boldsymbol{f}_M) = 0$$

defines then $f_M$ as an algebraic function of these indeterminates, and the higher coefficients $f_{M+1}, f_{M+2}, f_{M+3}, \cdots$ become rational functions of $f_0, f_1, \cdots, f_M$. It can in fact be proved that

$$(15) \qquad f_{M+h} = \frac{H_h(f_0, f_1, \cdots, f_M)}{(M+h)!\, G_M(f_0, 1!f_1, \cdots, M!f_M)^{2h-1}} \qquad (h = 1, 2, 3, \cdots).$$

Here the $H_h$ are polynomials in $f_0, f_1, \cdots, f_M$ at most of the total degree

$$c_1 h,$$

and with rational integral coefficients at most of the absolute value

$$c_2^h \cdot h!.$$

Here $c_1$ and $c_2$ are two positive constants which do not depend on $h$.

The solution $f$ of (G) so defined is of maximum dimension M and belongs to the suffix set $\{0, 1, \cdots, M-1\}$. On substituting for $f_0, f_1, \cdots, f_{M-1}$ any special complex or $p$-adic values $f_0, f_1, \cdots, f_{M-1}$, respectively, determining $f_M$ from

$$G(f_0, 1!f_1, \cdots, M!f_M) = 0,$$

and assuming that

$$G_M(f_0, 1!f_1, \cdots, M!f_M) \neq 0,$$

we obtain a determinate solution of (G). By the estimates for the degree and the height of $H_h$, this formal power series is in fact *convergent* if $|z|$ in the complex case and $|z|_p$ in the $p$-adic case is not too large.

Since (G) does not involve $z$ explicitly, exactly analogous results hold in the neighbourhood of any other complex or $p$-adic point $z = c$. We may consider simultaneously these solutions for all valuations of $Q$. There is thus something like a global theory of the manifold M, which involves also the analytic mappings of the neighbourhoods of different points $z = 0$ and $z = c$ into each other.

## References


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