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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Serre duality on complex analytic spaces.* Nota di ALDO ANDREOTTI e ARNOLD KAS, presentata (*) dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Si stabilisce un teorema di dualità pei gruppi di comologia sugli spazi analitici complessi.

In [8], J. P. Serre proved a duality theorem for locally free sheaves on a complex analytic manifold. This theorem has been generalized by Malgrange [5] to the case of an arbitrary coherent analytic sheaf using his results on the division of distributions. An elementary proof has also been given by Suominen [9]. In the algebraic case, the duality theorem has been proved by Grothendieck (cf. [4]), who also proved a relative version for proper morphisms $f: X \to Y$. In this note, we outline a proof of the duality theorem, in the absolute case, for coherent analytic sheaves on complex analytic spaces. For simplicity, we state our theorem for compact spaces, and we indicate the corresponding results for non-compact spaces in our concluding remarks. Full details will appear elsewhere [1] (1).

(*) Nella seduta del 17 aprile 1971.

(1) *Added in Proof.*

The duality theorem for complex analytic spaces has appeared in both the absolute case and relative case, due to Ramis and Ruget, and Ramis, Ruget, and Verdier respectively. (Cf. Publications Math. I.H.E.S. No. 38 (1970)). We feel that our version of the theorem and our proof are sufficiently different to warrant a separate publication.

1. *Statement of the Theorem.* Let X be a compact complex analytic space, and let $\mathcal{F}$ be a coherent analytic sheaf on X. We will define a sequence of coherent analytic sheaves, called dualizing sheaves, $\mathcal{D}^p \mathcal{F}$, $p \geq 0$.

THEOREM: *There exists a spectral sequence,* $\{E_r^{p,q}\}_{p \leq 0, q \geq 0}$ *with initial terms*:

$$E_2^{p,q} = H^{-p}(X, \mathcal{D}^q \mathcal{F})$$

which converges to the dual of the space:

$$H^{p+q}(X, \mathcal{F}).$$

2. *Definition of the Dualizing Sheaves.* Let Y be a Stein manifold, and let $\mathcal{F}$ be a coherent analytic sheaf on Y. We wish to define a topology on the cohomology groups with compact support $H_c^p(Y, \mathcal{F})$. We choose a countable covering $\{V_j\}$ of Y by relatively compact Stein subspaces V_j. Then $H_c^*(Y, \mathcal{F})$ is isomorphic to the cohomology of the Cech complex

$$\coprod_{j_0 \cdots j_p} \Gamma(V_{j_0 \cdots j_p}, \mathcal{F}).$$

The terms of this complex are direct sums of Fréchet spaces, hence they induce a topology on $H_c^*(Y, \mathcal{F})$.

LEMMA 1:

(*a*) The topology on $H_c^*(Y, \mathcal{F})$ is independent of the choice of the covering $\{V_j\}$.

(*b*) If $\mathcal{F}$ is a locally free sheaf, then the above topology on $H_c^*(Y, \mathcal{F})$ is identical to the topology defined by the "Dolbeault" resolution of differential forms (or currents) with compact supports.

(*c*) $H_c^*(Y, \mathcal{F})$ is the strong dual of a space of Fréchet-Schwartz (for the definition, cfr. [7], page 111).

The dualizing sheaf $\mathcal{D}^q \mathcal{F}$ is defined by a presheaf $D^q \mathcal{F}$. We define $D^q \mathcal{F}$ on each open set $U \subset X$ such that U is isomorphic to an analytic subspace of a Stein manifold.

DEFINITION 2:

$$D^q \mathcal{F}(U) = H_c^q(U, \mathcal{F})'$$

= the topological dual of $H_c^q(U, \mathcal{F})$.

If $U \subset W$, the restriction

$$D^q \mathcal{F}(W) \to D^q \mathcal{F}(U)$$

is defined to be the transpose of the "inclusion" mapping:

$$H_c^q(U, \mathcal{F}) \to H_c^q(W, \mathcal{F}).$$

The main point is to prove the coherence of $\mathcal{D}^q \mathcal{F}$. As this is a local question, we may assume that X is imbedded in a small polydisk $W \subset \mathbf{C}^m$, and by extending by zero, we may assume that $\mathcal{F}$ is defined on W. Moreover, we may assume that $\mathcal{F}$ admits a free resolution:

$$\cdots \to \mathcal{O}_W^{l_k} \to \cdots \to \mathcal{O}_W^{l_1} \to \mathcal{O}_W^{l_0} \to \mathcal{F} \to 0\,.$$

Let $\Omega\,(\cong \mathcal{O}_W)$ be the sheaf of germs of holomorphic m–forms on W. By applying $\mathcal{HOM}(\cdot\,,\Omega)$ to the above sequence, we get the sequence of sheaves:

$$\cdots \to \Omega^{l_k} \leftarrow \cdots \leftarrow \Omega^{l_1} \leftarrow \Omega^{l_0} \leftarrow \mathcal{HOM}(\mathcal{F}\,,\Omega) \leftarrow 0\,.$$

To each Stein open set $Z \subset W$, we consider the following complexes of topological vector spaces:

$$(*) \qquad \cdots \to H_c^m(Z\,,\mathcal{O}_W^{l_k}) \to \cdots \to H_c^m(Z\,,\mathcal{O}_W^{l_0})$$

$$(**) \qquad \cdots \leftarrow \Gamma(Z\,,\Omega^{l_k}) \leftarrow \cdots \leftarrow \Gamma(Z\,,\Omega^{l_0})\,.$$

By Serre duality on Z, it follows that the complexes (*) and (**) are topological dual to one another. As the mappings of (**) are topological homomorphisms, it follows from the duality lemma (cf. e.g. [6], page 214–08) that the homology groups of (*) and the cohomology groups of (**) are topologically dual. Now using the fact that $H_c^p(Z\,,\mathcal{O}_W) = 0$ for $p < n$ ([8]), it follows from an easy spectral sequence that the $m-q$'th homology group of (*) is isomorphic to $H_c^q(Z\,,\mathcal{F})$. It can be shown that this is, in fact, a topological isomorphism. It follows that $D^q\mathcal{F}(Z) = H_c^q(Z\,,\mathcal{F})'$ is equal to the $m-q$'th cohomology group of (**). Thus (cf. [3])

$$D^q\mathcal{F}(Z) = \mathrm{Ext}^{m-q}(Z\,;\mathcal{F}\,,\Omega)\,.$$

It follows that:

$$\mathcal{D}^q\mathcal{F} = \mathcal{EXT}_{\mathcal{O}_W}^{m-q}(\mathcal{F}\,,\Omega)$$

is a coherent analytic sheaf.

3. *Proof of the Theorem*. To each open set $U \subset X$, we may define a sheaf $\mathcal{F}_U$ such that:

$$\Gamma(X\,,\mathcal{F}_U) = \{\sigma \in \Gamma(X\,,\mathcal{F}) \mid \text{support } \sigma \subset U\} \qquad \text{(cf. [2], p. 139).}$$

In particular if $U \subset X$ is relatively compact, then $\Gamma(X\,,\mathcal{F}_U) = \Gamma_c(U, \mathcal{F})$. It follows that:

$$H^q(X\,,\mathcal{F}_U) = H_c^q(U\,,\mathcal{F})\,.$$

Now let $\{U_\alpha\}$ be a finite covering of X by open subsets U_α, such that each U_α may be imbedded in a Stein manifold. We define:

$$\mathcal{F}_{\alpha_0\cdots\alpha_p} = \mathcal{F}_U\,, \qquad \text{where} \quad U = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}\,.$$

There exists an exact sequence of sheaves on X:

$$\cdots \to \Pi \mathcal{F}_{\alpha_0 \cdots \alpha_p} \to \cdots \to \Pi \mathcal{F}_\alpha \to \mathcal{F} \to 0 .$$

This determines a spectral sequence $\{E_r^{p,q}\}$ converging to $H^*(X, \mathcal{F})$ for which

$$E_1^{p,q} = H^q(X, \Pi \mathcal{F}_{\alpha_0 \cdots \alpha_p}) = \Pi H_c^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{F}) .$$

If we give each $E_1^{p,q}$ the topology induced by $H_c^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{F})$, we find that the complex $E_1^{p,q}$ is the topological dual of the complex of Cech cochains of $\mathcal{D}^q \mathcal{F}$ with respect to the acyclic covering $\{U_\alpha\}$. Since the cohomology groups $H^*(X, \mathcal{D}^q \mathcal{F})$ are finite dimensional, it follows from the duality lemma that $H^*(X, \mathcal{D}^q \mathcal{F})$ is the dual of $E_2^{p,q}$. Our result now follows by "dualizing" the spectral sequence $\{E_r^{p,q}\}$.

4. *Remarks*.

(a) If X is non-singular, the construction of § 2 shows that $\mathcal{D}^q \mathcal{F}$ is the sheaf $\mathcal{E}\mathcal{X}\mathcal{T}^{n-q}(\mathcal{F}, \Omega)$. In this case, we get Serre duality in the form of [5] or [9].

(b) A refinement of our arguments here leads to a result in the non-compact case. Thus let Φ, Ψ be two paracompactifying families of supports which are dual in the following sense:

$$\Phi = \{C \subset X, \text{ closed} \mid C \cap A \text{ is compact } \forall A \in \Psi\}$$
$$\Psi = \{C \subset X, \text{ closed} \mid C \cap B \text{ is compact } \forall B \in \Phi\} .$$

(e.g., the families of closed supports and compact supports).

Then there exists a spectral sequence of topological vector spaces $\{E_r^{p,q}\}_{p \leq 0, q \leq 0}$ with

$$E_2^{p,q} = H_\Phi^{-p}(X, \mathcal{D}^q \mathcal{F}) .$$

Theorem: Assume that each differential

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1} \qquad r \geq r_0$$

is a topological homomorphism with closed range. Then the spectral sequence $\{E_r^{p,q}\}$ converges to the topological dual of

$$H_\Psi^{p+q}(X, \mathcal{F}) .$$

(c) The arguments involved in the proof of the duality theorem lead naturally to the notion of Cech homology groups with coefficients in a "cosheaf". This approach is followed in [1].

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