Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## On Almansi-Michell's problem for anisotropic beams

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **50** (1971), n.3, p. 299–303.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_50_3_299_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1971.

**Meccanica.** — *On Almansi–Michell's problem for anisotropic beams.* Nota di CONSTANTIN I. BORŞ, presentata (*) dal Socio D. GRAFFI.

RIASSUNTO. — Si da una soluzione per il problema di Almansi–Michell definito da (2) e (3) nel caso in cui la legge di Hooke è espressa dalle (4a) e (4b). La presente soluzione qui è più semplice di quella già conosciuta.

Let us consider a cylindrical beam limited by two planes $x_3 = 0$, $x_3 = h$ and by a cylindrical surface $\mathcal{F}$.

The domain occupied by the beam will be denoted by $\mathcal{V}$.

The domain of a cross-section of the beam and its area will be denoted by S and the boundary of S by $\Gamma$.

We take the axis of $x_3$ to be the central-line of the beam and the axes of $x_1$ and $x_2$ to be the principal axes of inertia of the end $x_3 = 0$. In this case we have

$$(1) \qquad \iint_S x_1 \, d\sigma = 0 \quad , \quad \iint_S x_2 \, d\sigma = 0 \quad , \quad \iint_S x_1 x_2 \, d\sigma = 0 .$$

We suppose that there are no body forces. It follows that the stress components $\sigma_{ij}$ must satisfy the equilibrium equations

$$(2) \qquad \sigma_{ij,j} = 0 \quad \text{in } \mathcal{V}.$$

We suppose also that the tractions applied on the lateral surface are such that

$$(3) \qquad \sigma_{ij} n_j = \tau_i (x_1 , x_2) \quad \text{on } \mathcal{F},$$

where $n_i$ are the direction cosines of the exterior normal to the surface $\mathcal{F}$ and $\tau_i (x_1 , x_2)$ are given functions of $x_1$ and $x_2$.

At the ends we will apply tractions such as to equilibrate the loads (3).

In the above formulae we used the summation convention over the repeated indices. The index $j$ after comma indicates partial differentiation with respect to $x_j$.

Here, we will find a solution which satisfies the equations (2) and (3). After that it remains to satisfy the end conditions; this is another problem and we know how to solve it.

The problem defined by (2) and (3) was solved by Almansi [1] and Michell [2] in the isotropic case.

Khatiashvili called the problem defined by (2) and (3) – Almansi–Michell's problem—and he solved it for the orthotropic case [3] and for the case when the material of the beam is anisotropic with one plane of elastic symmetry [4].

(*) Nella seduta del 13 marzo 1971.

In another paper a solution of Almansi-Michell's problem was given for the orthotropic case [5], the given solution being simpler than Khatiashvili's solution.

In this Note we will give a generalization of our results (see [5]) to the case when the material of the beam is anisotropic with one plane of elastic symmetry perpendicular to the $x_3$–axis so that the relations between the strain components $\gamma_{ij}$ and the stress components $\sigma_{ij}$ could be written in the form [6]

$$(4a)\quad \begin{cases} \gamma_{11} = \frac{1}{E}(\nu_{11}\sigma_{11} + \nu_{12}\sigma_{22} + \nu_{13}\sigma_{12} - \nu_1\sigma_{33}), \\ \gamma_{22} = \frac{1}{E}(\nu_{12}\sigma_{11} + \nu_{22}\sigma_{22} + \nu_{23}\sigma_{12} - \nu_2\sigma_{33}), \\ \gamma_{33} = \frac{1}{E}(-\nu_1\sigma_{11} - \nu_2\sigma_{22} - \nu_3\sigma_{12} + \sigma_{33}); \end{cases}$$

$$(4b)\quad \gamma_{23} = \frac{1}{LM - N^2}(M\sigma_{23} - N\sigma_{31}) \quad , \quad \gamma_{31} = \frac{-1}{LM - N^2}(N\sigma_{23} - L\sigma_{31}),$$

where $\nu_{ij}, \nu_i, L, M, N$ are constants which characterize the elastic properties of the material of the beam.

In order to solve the Almansi-Michell's problem let us suppose that

$$(5)\quad \begin{cases} \sigma_{11} = \frac{\partial^2\Phi}{\partial x_2^2} - M(\omega_1 + \theta_1) - N\int \frac{\partial\theta_2}{\partial x_2} dx_1, \\ \sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2} - L(\omega_1 + \theta_2) - N\int \frac{\partial\theta_1}{\partial x_1} dx_2, \\ \sigma_{12} = -\frac{\partial^2\Phi}{\partial x_1\,\partial x_2} - N\omega_1 \;,\; \sigma_{33} = \Omega + E\left[A_0 x_3 + \frac{1}{2}(A_1 x_1 + A_2 x_2)x_3^2\right]; \\ \sigma_{23} = L\frac{\partial}{\partial x_2}[(\omega_1 + \theta_2)x_3 + \omega_0] + N\frac{\partial}{\partial x_1}[(\omega_1 + \theta_1)x_3 + \omega_0], \\ \sigma_{31} = N\frac{\partial}{\partial x_2}[(\omega_1 + \theta_2)x_3 + \omega_0] + M\frac{\partial}{\partial x_1}[(\omega_1 + \theta_1)x_3 + \omega_0], \end{cases}$$

where $\omega_0, \omega_1, \Phi$ are unknown functions of $x_1$ and $x_2$,

$$(6)\quad \begin{cases} \Omega = E\left\{\omega_1 + \frac{1}{2}\left[\theta_1 + \theta_2 + \left(\frac{1}{3}\nu_1 x_1 + \frac{1}{2}\nu_3 x_2\right)A_1 x_1^2 + \right.\right. \\ \quad \left.\left. + \left(\frac{1}{3}\nu_2 x_2 + \frac{1}{2}\nu_3 x_1\right)A_2 x_2^2\right]\right\} + \nu_1\left[\frac{\partial^2\Phi}{\partial x_2^2} - M(\omega_1+\theta_1) - N\int\frac{\partial\theta_2}{\partial x_2}dx_1\right] + \\ \quad + \nu_2\left[\frac{\partial^2\Phi}{\partial x_1^2} - L(\omega_1 + \theta_2) - N\int\frac{\partial\theta_1}{\partial x_1}dx_2\right] + \nu_3\left[-\frac{\partial^2\Phi}{\partial x_1\,\partial x_2} - N\omega_1\right], \\ \theta_1 = b_1 x_1 x_2^2 - c_1 x_1 x_2 \quad , \quad \theta_2 = a_1 x_1^2 x_2 + c_1 x_1 x_2 \end{cases}$$

and $A_0, A_1, A_2, a_1, b_1, c_1$ are constants which must be determined.

The unknown functions and the constants introduced above must be determined such that all the equations and conditions we need be satisfied.

The stresses (5) will verify the equations of equilibrium if the functions $\omega_0$ and $\omega_1$ satisfy the equations

$$\Delta\,\omega_0 + \mathrm{EA}_0 = 0 \quad \text{in S} \tag{7a}$$

and

$$\Delta\omega_1 + (\mathrm{EA}_1 + 2\,\mathrm{N}a_1)\,x_1 + (\mathrm{EA}_2 + 2\,\mathrm{N}b_2)\,x_2 = 0 \quad \text{in S.} \tag{8a}$$

The third boundary condition (3) will be satisfied if the function $\omega_0$ and $\omega_1$ are chosen such that

$$\mathfrak{D}\omega_0 = \tau_3 \quad \text{on } \Gamma, \tag{7b}$$

$$\mathfrak{D}\omega_1 = -\frac{\partial}{\partial x_1}(\mathrm{N}\,\theta_2 + \mathrm{M}\,\theta_1)\,n_1 - \frac{\partial}{\partial x_2}(\mathrm{L}\,\theta_2 + \mathrm{N}\,\theta_1)\,n_2 = \tag{8b}$$
$$= -(\mathrm{N}n_1 + \mathrm{L}n_2)(a_1 x_1^2 + c_1 x_1) - (b_1 x_2^2 - c_1 x_2)(\mathrm{M}n_1 + \mathrm{N}n_2) \quad \text{on } \Gamma.$$

The above operators $\Delta$ and $\mathfrak{D}$ are given by

$$\Delta = \mathrm{M}\frac{\partial^2}{\partial x_1^2} + 2\,\mathrm{N}\frac{\partial^2}{\partial x_1\,\partial x_2} + \mathrm{L}\frac{\partial^2}{\partial x_2^2}, \tag{9}$$

$$\mathfrak{D} = n_1\left(\mathrm{M}\frac{\partial}{\partial x_1} + \mathrm{N}\frac{\partial}{\partial x_2}\right) + n_2\left(\mathrm{N}\frac{\partial}{\partial x_1} + \mathrm{L}\frac{\partial}{\partial x_2}\right). \tag{10}$$

The function $\omega_1$ exists with arbitrary $a_1, b_1, c_1$. The condition of existence of the function $\omega_0$ requires that

$$\mathrm{A}_0 = \frac{1}{\mathrm{ES}}\int_\Gamma \tau_3\,\mathrm{d}s.$$

Using (4) and (5) we can calculate the strains $\gamma_{ij}$ and they must satisfy the conditions of compatibility of Saint-Venant.

All compatibility conditions will be satisfied if

$$a_1 = \nu_1\,\mathrm{A}_2 - \frac{1}{2}\,\nu_3\,\mathrm{A}_1 \quad , \quad b_1 = \nu_2\,\mathrm{A}_1 - \frac{1}{2}\,\nu_3\,\mathrm{A}_2$$

and the function $\Phi$ verifies the equation

$$\beta_{22}\frac{\partial^4\Phi}{\partial x_1^4} - 2\,\beta_{23}\frac{\partial^4\Phi}{\partial x_1^3\,\partial x_2} + (2\,\beta_{12} + \beta_{33})\frac{\partial^4\Phi}{\partial x_1^2\,\partial x_2^2} - 2\,\beta_{13}\frac{\partial^4\Phi}{\partial x_1\,\partial x_2^3} + \tag{11}$$
$$+ \beta_{11}\frac{\partial^4\Phi}{\partial x_2^4} = (\mathrm{M}\beta_{12} + \mathrm{L}\beta_{22} + \mathrm{N}\beta_{23} + \nu_2)\frac{\partial^2\omega_1}{\partial x_1^2} -$$
$$-(\mathrm{M}\beta_{13} + \mathrm{L}\beta_{23} + \mathrm{N}\beta_{33} + \nu_3)\frac{\partial^2\omega_1}{\partial x_1\,\partial x_2} + (\mathrm{M}\beta_{11} + \mathrm{L}\beta_{12} + \mathrm{N}\beta_{13} + \nu_1)\frac{\partial^2\omega_1}{\partial x_2^2} +$$
$$+ 2[(\mathrm{M}\beta_{11} + \nu_1)\,b_1 x_1 + \mathrm{L}(\beta_{22} + \nu_2)\,a_1 x_2] + (\mathrm{N}\beta_{12} - \mathrm{L}\beta_{23})(2\,a_1 x_1 + c_1) +$$
$$+ (\mathrm{N}\beta_{12} - \mathrm{M}\beta_{13})(2\,b_1\,x_2 - c_1) \quad \text{in S,}$$

where

$$\beta_{ij} = \frac{\nu_{ij} - \nu_i\,\nu_j}{\mathrm{E}} \qquad (i, j = 1, 2, 3).$$

From the first two equations (3) we get the following boundary conditions for the function $\Phi$:

$$(12) \quad \begin{cases} \dfrac{\partial^2 \Phi}{\partial x_2^2} n_1 - \dfrac{\partial^2 \Phi}{\partial x_1 \partial x_2} n_2 = (\mathrm{M} n_1 + \mathrm{M} n_2)\, \omega_1 + \\ \qquad + \left[\mathrm{M}\,(b_1 x_2 - c_1)\, x_1 x_2 + \mathrm{N}\left(\frac{1}{3} a_1 x_1 + \frac{1}{2} c_1\right) x_1^2\right] n_1 + \tau_1\,, \\ -\dfrac{\partial^2 \Phi}{\partial x_1 \partial x_2} n_1 + \dfrac{\partial^2 \Phi}{\partial x_1^2} n_2 = (\mathrm{N} n_1 + \mathrm{L} n_2)\, \omega_1 + \\ \qquad + \left[\mathrm{L}\,(a_1 x_1 + c_1)\, x_1 x_2 + \mathrm{N}\left(\frac{1}{3} b_1 x_2 - \frac{1}{2} c_1\right) x_2^2\right] n_2 + \tau_2 \quad \text{on } \Gamma. \end{cases}$$

It is obvious that we can obtain the function $\Phi$ in a similar way to that corresponding to Airy's function in the plane problem of anisotropic bodies with one plane of elastic symmetry.

Taking into account the fact that for any function $\psi$ which satisfies the equation $\Delta\psi = 0$ we have

$$\int_\Gamma \omega_1 \mathfrak{D}\psi \, ds = \int_\Gamma \psi \mathfrak{D}\omega_1 \, ds - \iint_S \psi \Delta\omega_1 \, d\sigma\,,$$

we can write down the following formulae

$$\int_\Gamma (\mathrm{M} n_1 + \mathrm{N} n_2)\, \omega_1 \, ds = \int_\Gamma x_1 \mathfrak{D}\omega_1 \, ds - \iint_S x_1 \Delta\omega_1 \, d\sigma\,,$$

$$\int_\Gamma (\mathrm{N} n_1 + \mathrm{L} n_2)\, \omega_1 \, ds = \int_\Gamma x_2 \mathfrak{D}\omega_1 \, ds - \iint_S x_2 \Delta\omega_1 \, d\sigma\,,$$

$$\int_\Gamma [(\mathrm{M} x_2 - \mathrm{N} x_1)\, n_1 + (\mathrm{N} x_2 - \mathrm{L} x_1)\, n_2]\, \omega_1 \, ds = \int_\Gamma \varphi \mathfrak{D}\omega_1 \, ds - \iint_S \varphi \Delta\omega_1 \, d\sigma,$$

where $\varphi$ is the function of torsion defined by

$$\Delta\varphi = 0 \quad \text{in S},$$

$$\mathfrak{D}\varphi = (\mathrm{M} x_2 - \mathrm{N} x_1)\, n_1 + (\mathrm{N} x_2 - \mathrm{L} x_1)\, n_2 \quad \text{on } \Gamma.$$

Using the above formulae we can determine the constants $A_1, A_2, c_1$ before to know the function $\omega_1$.

Thus, from the conditions of existence of function $\Phi$ we get

$$A_1 = -\frac{1}{EI_2} \int_\Gamma \tau_1 \, ds \quad , \quad A_2 = -\frac{1}{EI_1} \int_\Gamma \tau_2 \, ds\,,$$

$$c_1 = \frac{1}{D_t} \left\{ \iint_S \left[ (EA_1 x_1 + EA_2 x_2)\, \varphi - (\mathrm{L} a_1 x_1^2 + \mathrm{N} b_1 x_2^2) \left( \frac{\partial\varphi}{\partial x_2} + x_1 \right) - \right.\right.$$

$$\left.\left. - (\mathrm{N} a_1 x_1^2 + \mathrm{M} b_1 x_2^2) \left( \frac{\partial\varphi}{\partial x_1} - x_2 \right) \right] d\sigma + \int_\Gamma (\tau_1 x_2 - \tau_2 x_1)\, ds \right\},$$

where

$$I_i = \iint_S x_j^2 \, d\sigma \qquad (i, j = 1, 2 \quad ; \quad i \neq j),$$

$$D_t = \iint_S \left[ L x_1^2 + M x_2^2 - 2 N x_1 x_2 + L x_1 \frac{\partial \varphi}{\partial x_2} - M x_2 \frac{\partial \varphi}{\partial x_1} - N \left( x_2 \frac{\partial \varphi}{\partial x_2} - x_1 \frac{\partial \varphi}{\partial x_1} \right) \right] d\sigma .$$

$D_t$ is the torsional rigidity of the beam and it is always a positive number [6].

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