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Analisi matematica. — *On the solution of a non-linear mixed problem for the Navier-Stokes equations in a time dependent domain.* Nota I di GIOVANNI PROUSE [(*)], presentata [(**)] dal Corrisp. L. AMERIO.

RIASSUNTO. — Si considera, per le equazioni di Navier-Stokes in un dominio bidimensionale dipendente dal tempo, un problema misto con condizioni al contorno non lineari e si enunciano un teorema di esistenza ed unicità della soluzione e tre teoremi ausiliari, le cui dimostrazioni vengono date nelle successive Note II e III.

1. INTRODUCTION AND STATEMENTS

In the present note and in the following two we shall again consider the mixed problem for the Navier-Stokes equations studied in [1], assuming however that the domain in which the motion of the fluid takes place depends on the time t with a given law: $\Omega = \Omega(t) = \Omega_t$.

Let, precisely, Ω_t be an open, bounded set of the x_1, x_2 plane, depending on t and let $\Gamma(t) = \Gamma_t$ be its boundary, which we shall assume is constituted by the lines

$$\Gamma_1 = \{x_1 = 0 \quad , \quad k_1' \leq x_2 \leq k_2'\}$$
$$\Gamma_2 = \{x_1 = l \quad , \quad k_1'' \leq x_2 \leq k_2''\}$$
$$\Gamma_{3,t} = \{0 \leq x_1 \leq l \quad , \quad x_2 = \psi_i(x_1, t) \, , \quad (i = 1, 2)\} .$$

As is well known, the motion in Ω_t of an incompressible fluid of viscosity μ and density 1 subject to the external force $\vec{f}(x, t) = \{f_1(x, t), f_2(x, t)\}$ $(x = (x_1, x_2))$ is governed by the equations

$$(1.1) \qquad \begin{cases} \dfrac{\partial u_j}{\partial t} - \mu\Delta u_j + \displaystyle\sum_{k=1}^{2} u_k \frac{\partial u_j}{\partial x_k} + \frac{\partial p}{\partial x_j} = f_j & (j = 1, 2) \\ \displaystyle\sum_{k=1}^{2} \frac{\partial u_k}{\partial x_k} = 0 \end{cases}$$

where $\vec{u}(x, t) = \{u_1(x, t), u_2(x, t)\}$ denotes the velocity and $p(x, t)$ the pressure. Denoting by $\vec{\nu}_t$ the outside normal to Γ_t, we shall consider the boundary conditions defined by the relations

$$(1.2) \qquad \begin{cases} \dfrac{1}{2} u_1^2(x, t) + p(x, t) = \alpha_i(x, t) & (x \in \Gamma_i \, , \; 0 \leq t \leq T \, , \; i = 1, 2) \\ p(x, t) = \beta(x, t) \, \vec{u}(x, t) \times \vec{\nu}_t \, |\vec{u}(x, t) \times \vec{\nu}_t| & (x \in \Gamma_{3,t} \, , \; 0 \leq t \leq T) \\ |\vec{u}(x, t)| = |\vec{u}(x, t) \times \vec{\nu}_t| & (x \in \Gamma_t \, , \; 0 \leq t \leq T) . \end{cases}$$

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As explained in greater detail in [1], the boundary conditions (1.2) assign the value of the "total energy" $\frac{1}{2}|\vec{u}|^2 + p$ of the fluid on the initial and final sections, Γ_1 and Γ_2, of the "tube" Ω_t and express the relationship between the pressure and the velocity of the fluid through the "wall" $\Gamma_{3,t}$, which is assumed to be permeable.

From experimental data it appears, in fact, that the velocity of the fluid through a permeable wall is orthogonal to the wall and is proportional to the square root of the jump of pressure. In the second of (1.2) it has obviously been assumed that the external pressure is zero.

Finally, the third of (1.2) interprets the condition that along Γ_t the component of the velocity tangent to Γ_t vanishes; this follows, on $\Gamma_{3,t}$, from the limit layer theory and, on Γ_1 and Γ_2, from the assumption that, on the initial and final sections, the velocity $\vec{u}$ coincides with its normal component.

Our aim is to give an existence and uniqueness theorem of the solution of equations (1.1) satisfying the boundary conditions (1.2) and the initial condition

$$\vec{u}(x, 0) = \vec{u}_0(x) \qquad (x \in \Omega_0) . \tag{1.3}$$

Let us begin by giving some definitions and basic notations.

Let Ω be an open set of the (x_1, x_2) plane satisfying the cone property and denote by $\mathfrak{N}(\Omega)$ the manifold of vectors $\vec{v}(x) = \{v_1(x), v_2(x)\}$ indefinitely differentiable in Ω, with null divergence and such that $|\vec{v}(x)| = |\vec{v}(x) \times \vec{\nu}|$ when $x \in \Gamma$ (boundary of Ω). Further, denote by $N^\sigma(\Omega)$ the closure of $\mathfrak{N}(\Omega)$ in $H^\sigma(\Omega)$ and observe that, by the definitions given, we can set

$$\begin{aligned} (\vec{v}, \vec{w})_{N^0(\Omega)} &= (\vec{v}, \vec{w})_{L^2(\Omega)} = \int_\Omega \sum_{j=1}^{2} v_j(x)\, w_j(x)\, d\Omega \\ (\vec{v}, \vec{w})_{N^1(\Omega)} &= (\vec{v}, \vec{w})_{H_0^1(\Omega)} = \int_\Omega \sum_{j,k=1}^{2} \frac{\partial v_j}{\partial x_k} \frac{\partial w_j}{\partial x_k}\, d\Omega , \end{aligned} \tag{1.4}$$

since, for functions belonging to $\mathfrak{N}(\Omega)$, the H_0^1– and H^1–norms are equivalent; moreover $N^1(\Omega)$ is dense in $N^0(\Omega)$.

Let D(A) be the set of elements $\vec{u} \in N^1(\Omega)$ such that the linear form $\vec{v} \to (\vec{u}, \vec{v})_{N^1(\Omega)}$ is continuous in the topology of $N^0(\Omega)$; it is then possible to define a linear, self-adjoint, positive operator A, from D(A) to $N^0(\Omega)$, such that

$$(\vec{u}, \vec{v})_{N^1(\Omega)} = (A\vec{u}, \vec{v})_{N^0(\Omega)} \qquad \forall \vec{u} \in D(A) , \quad \vec{v} \in N^1(\Omega) . \tag{1.5}$$

Let us denote by A^σ (σ real ≥ 0) the power of order σ of A and by $V_\sigma(\Omega) = D(A^{\sigma/2})$ the domain of $A^{\sigma/2}$; $V_\sigma(\Omega)$ is a Hilbert space with scalar product defined by

$$(\vec{u}, \vec{v})_{V_\sigma(\Omega)} = (A^{\sigma/2}\vec{u}, A^{\sigma/2}\vec{v})_{N^0(\Omega)} . \tag{1.6}$$

We have $N^1(\Omega) = V_1(\Omega)$, $N^0(\Omega) = V_0(\Omega)$, $H^\sigma(\Omega) \supset V_\sigma(\Omega)$ and the interpolation property holds

$$[V_\alpha(\Omega), V_\beta(\Omega)]_\theta = V_{\alpha(1-\theta)+\beta\theta}(\Omega), \tag{1.7}$$

where $[\mathcal{H}_1, \mathcal{H}_2]_\theta$ ($\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces) indicates the Hilbert space $\mathcal{H}_1^{1-\theta}\mathcal{H}_2^\theta$ which is intermediate between $\mathcal{H}_1$ and $\mathcal{H}_2$ according to the definition given by Lions [2]. Identifying $V_0(\Omega)$ with its dual $V_0'(\Omega)$, we can define $V_{-\sigma}(\Omega)$ by setting $V_\sigma'(\Omega) = V_{-\sigma}(\Omega)$. We now give the definition of solution of equations (1.1), (1.2), to which we shall always refer in what follows.

Assuming that $\vec{f}(t) = \{\vec{f}(x,t); x \in \Omega_t\} \in L^2(0, T; V_{\sigma-1}(\Omega_t))$, $\alpha_i(t) = \{\alpha_i(x,t); x \in \Gamma_i\} \in L^2(0,T; L^2(\Gamma_i))$, $\beta(t) = \{\beta(x,t); x \in \Gamma_{3,t}\} \in L^\infty(0,T; L^\infty(\Gamma_{3,t}))$, $\psi_i \in C^1$ and setting

$$b(t, \vec{u}, \vec{v}, \vec{w}) = \int_{\Omega_t} \sum_{i,j=1}^{2} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x)\, d\Omega_t,$$

we shall say that $\vec{u}(t) = \{\vec{u}(x,t); x \in \Omega_t\}$ *is a solution in* $[0,T]$ *of equations* (1.1) *satisfying the boundary conditions* (1.2) *if* (1):

a) $\vec{u}(t) \in L^2(0, T; V_{\sigma+1}(\Omega_t)) \cap L^\infty(0,T; V_\sigma(\Omega_t)) \cap H^1(0,T; V_{\sigma-1}(\Omega_t))$;

b) $\vec{u}(t)$ *satisfies,* $\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$, *the equation*

$$\begin{aligned}\int_0^T \{\langle \vec{u}'(t), \vec{h}(t)\rangle + \mu\langle A\vec{u}(t), \vec{h}(t)\rangle + \\ + b(t, \vec{u}(t), \vec{u}(t), \vec{h}(t)) - \langle \vec{f}(t), \vec{h}(t)\rangle\}\, dt = \\ = -\int_0^T \Big\{ \sum_{i=1}^{2} \int_{\Gamma_i} (\alpha_i(x,t) - \tfrac{1}{2} u_1^2(x,t))\, \vec{h}(x,t) \times \vec{\nu}_t\, d\Gamma_i + \\ + \int_{\Gamma_{3,t}} \beta(x,t)\, \vec{u}(x,t) \times \vec{\nu}_t\, |\vec{u}(x,t) \times \vec{\nu}_t|\, \vec{h}(x,t) \times \vec{\nu}_t\, d\Gamma_{3,t} \Big\}\, dt\end{aligned} \tag{1.8}$$

where $\langle\, ,\rangle$ denotes the duality between $V_{\sigma-1}(\Omega_t)$ and $V_{1-\sigma}(\Omega_t)$.

Relation (1.8) is obtained directly from the first of (1.1), multiplying it by a test function $h_j(t)$, integrating over $\Omega_t \times [0,T]$ and bearing in mind the second of (1.1) and the three conditions (1.2). It is, in fact, $\forall \vec{u}, \vec{h} \in V_1(\Omega_t)$, with $\Delta\vec{u} \in V_0(\Omega_t)$, by (1.4), (1.5),

$$-(\Delta\vec{u}, \vec{h})_{V_0(\Omega_t)} = -(\Delta\vec{u}, \vec{h})_{L^2(\Omega_t)} = (\vec{u}, \vec{h})_{H_0^1(\Omega_t)} = (\vec{u}, \vec{h})_{N^1(\Omega_t)} =$$
$$= (A\vec{u}, \vec{h})_{N^0(\Omega_t)} = (A\vec{u}, \vec{h})_{V_0(\Omega_t)},$$

(1) The meaning of the notations $L^p(0, T; V_\alpha(\Omega_t))$, etc. is obvious, bearing in mind the smoothness assumptions made on ψ_i.

since the integral $\int_{\Gamma_t} \frac{\partial \vec{u}}{\partial \nu_t} \times \vec{h}\, d\Gamma_t$ vanishes by the second of (1.1) and the third of (1.2). Moreover

$$\int_{\Omega_t} \sum_{j=1}^{2} \frac{\partial p}{\partial x_j} h_j \, d\Omega_t = \int_{\Gamma_t} p\vec{h} \times \vec{\nu}_t \, d\Gamma_t = \sum_{i=1}^{2} \int_{\Gamma_i} \left(\alpha_i - \frac{u_1^2}{2}\right) \vec{h} \times \vec{\nu}_t \, d\Gamma_i + \\ + \int_{\Gamma_{3,t}} \beta \vec{u} \times \vec{\nu}_t \, | \vec{u} \times \vec{\nu}_t | \, \vec{h} \times \vec{\nu}_t \, d\Gamma_{3,t} .$$

In the following paragraphs we shall prove some results from which will follow directly, by means of the Leray-Schauder principle, the existence of a solution (in the sense indicated above) of (1.1), (1.2), in a sufficiently small neighbourhood of $t = 0$, satisfying the initial condition

$$(1.9) \qquad \vec{u}(0) = \vec{u}_0 .$$

A uniqueness theorem of the solution will also be proved.

Such results are expressed by the following theorems which we here state.

THEOREM 1. *Assume that* $\vec{g}(t) \in L^2(0, T; V_{\sigma-1}(\Omega_t))$, $\vec{u}_0 \in V_\sigma(\Omega_0)$ *and that the functions* $\psi_i(x_1, t)$ *which define* $\Gamma_{3,t}$ *are continuous for* $0 \le x_1 \le l$, $0 \le t \le T$ *together with their derivatives* $\partial\psi_i/\partial t$, $\partial\psi_i/\partial x_1$. *There exists then, if* $0 \le \sigma < 1/2$, *a function* $\vec{u}(t) \in L^2(0, T; V_{\sigma+1}(\Omega_t)) \cap L^\infty(0, T; V_\sigma(\Omega_t)) \cap H^1(0, T; V_{\sigma-1}(\Omega_t))$ *satisfying,* $\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$, *the equation*

$$(1.10) \qquad \int_0^T \{\langle \vec{u}'(t), \vec{h}(t)\rangle + \mu \langle A\vec{u}(t), \vec{h}(t)\rangle - \langle \vec{g}(t), \vec{h}(t)\rangle\}\, dt = 0$$

and the initial condition (1.9).

Let us now denote by $S : \vec{u} = S(\vec{v})$ the transformation defined in the following way. Given any function $\vec{v}(t) \in L^2(0, T; V_{\sigma+1}(\Omega_t)) \cap L^\infty(0, T; V_\sigma(\Omega_t)) \cap H^1(0, T; V_{\sigma-1}(\Omega_t))$, we call $\vec{u}(t) = S\vec{v}(t)$ the function (which, as can easily be seen, exists, by Theorem 1) belonging to the same functional space as $\vec{v}(t)$ and satisfying (1.9) and the equation

$$(1.10) \qquad \int_0^T \{\langle \vec{u}'(t), \vec{h}(t)\rangle + \mu \langle A\vec{u}(t), \vec{h}(t)\rangle - \langle \vec{f}(t), \vec{h}(t)\rangle\}\, dt = \\ = -\int_0^T \Big\{ b(t, \vec{v}(t), \vec{v}(t), \vec{h}(t)) + \sum_{i=1}^{2} \int_{\Gamma_i} \left(\alpha_i(x,t) - \frac{v_1^2(x,t)}{2}\right) \vec{h}(x,t) \times \vec{\nu}_t \, d\Gamma_i + \\ + \int_{\Gamma_{3,t}} \beta(x,t)\, \vec{v}(x,t) \times \vec{\nu}_t \, | \vec{v}(x,t) \times \vec{\nu}_t | \, \vec{h}(x,t) \times \vec{\nu}_t \, d\Gamma_{3,t} \Big\}\, dt$$

$\forall \vec{h}(t) \in L^2(0, T; V_{1-\sigma}(\Omega_t))$.

We have then:

THEOREM 2. *If* $\vec{u}_0 \in V_\sigma(\Omega_0)$, $\vec{f}(t) \in L^2(0,T;V_{\sigma-1}(\Omega_t))$, $\alpha_i(t) \in L^2(0,T;L^2(\Gamma_i))$, $\beta(t) \in L^\infty(0,T;L^\infty(\Gamma_{3,t}))$ *and* $\partial\psi_i/\partial x_1$, $\partial\psi_i/\partial t$ *are continuous, then, for* $1/4 < \sigma < 1/2$, *the transformation* S *is completely continuous from* $W_{T,\sigma} = L^2(0,T;V_{\sigma+1}(\Omega_t)) \cap L^\infty(0,T;V_\sigma(\Omega_t)) \cap H^1(0,T;V_{\sigma-1}(\Omega_t))$ *in itself.*

Consider now the transformation, depending continuously on the parameter $\lambda \in [0,1]$, $\vec{u} = S(\vec{v}, \lambda)$ defined in the following way. Given any function $\vec{v}(t) \in W_{T,\sigma}$, we call $\vec{u}(t) = S(\vec{v}(t), \lambda)$ a function belonging to $W_{T,\sigma}$ and satisfying the conditions

$$\vec{u}(0) = \lambda \vec{u}_0 \tag{1.11}$$

$$\int_0^T \{\langle \vec{u}'(t), \vec{h}(t)\rangle + \mu \langle A\vec{u}(t), \vec{h}(t)\rangle - \lambda \langle \vec{f}(t), \vec{h}(t)\rangle\} dt = \tag{1.12}$$

$$= -\lambda \int_0^T \Big\{ b(t, \vec{v}(t), \vec{v}(t), \vec{h}(t)) + \sum_{i=1}^{2} \int_{\Gamma_i} \left(\alpha_i(x,t) - \frac{v_1^2(x,t)}{2}\right) \vec{h}(x,t) \times \vec{\nu}_t \, d\Gamma_i +$$

$$+ \int_{\Gamma_{3,t}} \beta(x,t)\, \vec{v}(x,t) \times \vec{\nu}_t \, |\vec{v}(x,t) \times \vec{\nu}_t| \, \vec{h}(x,t) \times \vec{\nu}_t \, d\Gamma_{3,t} \Big\} dt,$$

$\forall \vec{h}(t) \in L^2(0,T;V_{1-\sigma}(\Omega_t))$.

By the definitions given, it is obvious that $S(\vec{v}, 1) = S(\vec{v})$.

The following theorem holds:

THEOREM 3. *Suppose that all the assumptions of Theorem 2 are verified and let* $\vec{u}(t)$ *be any solution of the equation* $\vec{u}(t) = S(\vec{u}(t), \lambda)$. *Then if* T *is sufficiently small,* $\vec{u}(t)$ *is bounded on* $[0,T]$ *uniformly with respect to* λ, *i.e. there exists a constant* M_1, *independent of* $\vec{u}$ *and of* λ, *such that,* $\forall \lambda \in [0,1]$,

$$\|\vec{u}\|^2_{W_{T,\sigma}} = \int_0^T \|\vec{u}(t)\|^2_{V_{\sigma+1}(\Omega_t)} dt + \sup_{0 \le t \le T} \|\vec{u}(t)\|^2_{V_\sigma(\Omega_t)} + \int_0^T \|u'(t)\|^2_{V_{\sigma-1}(\Omega_t)} dt \le M_1 .$$

Moreover, $S(\vec{v}(t), 0) = 0$, *which means that equations* (1.11), (1.12), *written for* $\lambda = 0$, *admit the only solution* $\vec{u}(t) = 0$.

It is obvious that, by Theorems 2 and 3, the transformation S defined above satisfies the assumptions of the Leray-Schauder principle. Hence the functional equation $\vec{u}(t) = S(\vec{u}(t))$, which evidently corresponds to (1.8), (1.9), admits a solution $\vec{u}(t) \in W_{T,\sigma}$ and we obtain the following

THEOREM 4. *Under the assumptions made in Theorem 2, there exists in* $[0,T]$, *for* T *sufficiently small, a solution of equations* (1.1) *satisfying the initial condition* (1.3) *and the boundary conditions* (1.2), *with* $1/4 < \sigma < 1/2$.

We shall, finally, prove a uniqueness theorem for the solution thus obtained:

THEOREM 5. *There exists in* [0, T] (T *sufficiently small*) *at most one solution of* (1.1) *satisfying conditions* (1.2), (1.3).

The proofs of Theorems 1, 2, 3, 5 will be given in the following paragraphs.

Observation I. The theorems given can be extended, without any modification, to more general sets Ω_t than those considered. We may, for instance, assume that the " tube " branches, i.e. that its " wall " is constituted by the lines $x_2 = \psi_i'(x_1, t)$, $x_2 = \psi_i''(x_1, t)$ $(0 \leq x_1 \leq l,\ i = 1, \cdots, s)$ such that a parallel to x_2 directed as x_2 " enters " the tube through ψ_i' and " leaves " it through ψ_i''.

Observation II. The results obtained here will be utilized in the note (to appear on these Rendiconti) " On the motion of a viscous incompressible fluid in a tube with permeable and deformable wall ", in which we shall assume that the shape of the wall is not a given function of t, but depends on the pressure exercised by the fluid on the wall itself according to an appropriate law.

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