Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Right abelian categories

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.3, p. 284–289.

Accademia Nazionale dei Lincei

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Algebra. — *Right abelian categories.* Nota di SYED A. HUQ, presentata (*) dal Corrips. G. ZAPPA.

RIASSUNTO. — Si introducono le nozioni di categoria destra e di categoria abeliana destra, e se ne forniscono le proprietà fondamentali. Diversi risultati familiari sulle categorie abeliane seguono per dualità.

§ 1. – INTRODUCTION

The theory of near rings has seen rapid development in the last few years. The near rings which occur in nature [see Neumann [10]] differ from that of a ring by the fact that the addition need not be commutative and only one distributive law holds. The theory was further developed by Fröhlich [2], who studied the distributively generated ones and was able to show that these admit stronger structural properties. Both the theory of near rings and d.g. near rings was generalized topologically by Tharmaratnam [5], Beidleman and Cox [1] and by others; similar further generalizations such as near algebras (Brown [6]), near fields, (Zassenhaus [7]), came up.

In the following, we present a further generalization of the concept in category theoretic language in the same spirit, to examine how much we lose if we give up one of the distributive law, and abelianness of the hom sets in an additive category. Many familiar results on additive categories follow by duality. For basic definitions, notations etc, we refer our readers to Kurosh, Livshits and Schulgeifor [4], or Mitchell [3].

§ 2. RIGHT ADDITIVE CATEGORIES

A right additive category is a category $\mathcal{A}$, together with a group structure (written additively, not to imply commutativity) on each of its morphism sets [A , B] for objects A , B of $\mathcal{A}$, subject to the following conditions

(i) the composition function $[A, B] \times [B, C] \to [A, C]$ is right linear i.e. for $\alpha, \beta \in [A, B]$ and $\gamma \in [B, C]$ we have $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$.

(ii) The zero element of the group behaves as zero morphism i.e. $\alpha 0 = 0$, whenever the composition makes sense.

(iii) $\eta(-1) = -\eta$, whenever defined.

Thus the morphisms of a right additive category form a "near ringoid"; we note that for any object $A \in \mathcal{A}$, [A , A] is a near ring with 1_A called the *endomorphism near ring* of A. By Exercise 6, Ch. I of Mitchell [3], one can assume that such a category has zero object which we denote by 0.

(*) Nella seduta del 13 marzo 1971.

Examples

(i) All additive categories.

(ii) A right near ring with multiplicative identity is a right additive category with single object.

(iii) Consider the category K, whose objects are groups but morphisms are set functions fixing zero; addition and composition of morphisms are pointwise; K^{op} is right additive.

We note that the dual of a right additive category is left additive and a category which is both left and right additive is in fact an additive category [cfr. Proposition 2.2, below].

DEFINITION 2.1. We call a morphism $\gamma : A \to B$ *distributive in a right additive category* A, *with respect to the pair* $\alpha , \beta \in [B , C]$, if γ induces a homomorphism $[B , C] \xrightarrow{\gamma^*} [A , C]$ defined by $\gamma^*(\alpha) = \gamma\alpha$ of the groups on the sum $\alpha + \beta$. If the equation

$$\text{(H)} \qquad \gamma (\alpha + \beta) = \gamma\alpha + \gamma\beta$$

holds whenever this makes sense we call γ a distributive morphism of A.

Notice that in Exercise (iii) above, the dual of group homomorphisms are in fact distributive in K^{op}.

PROPOSITION 2.2. *If in a right additive category all morphisms are distributive, then it is an additive category.*

Proof. If all morphisms are distributive, then axiom II and III in the definition of right additive category is redundant and a consequence of equation (H). Next for two morphisms

$$\alpha , \beta \in [A , B]$$

$$(\alpha + \beta)(1_B + 1_B) = (\alpha + \beta) 1_B + (\alpha + \beta) 1_B = \alpha + \beta + \alpha + \beta$$

which is same as

$$\alpha (1_B + 1_B) + \beta (1_B + 1_B) = \alpha + \alpha + \beta + \beta$$

thus $\beta + \alpha = \alpha + \beta$. Hence $[A , B]$ is abelian.

Given any right additive category C, one can construct an additive category $\overline{C}$ in various ways. (A) Take objects $\overline{C}$ = objects C. For morphisms any one of (i). For $[A , B]_{\overline{C}}$, take the free abelian group generated by $[A , B]_C$ and then define the composition

$$\text{(M)} \qquad \left(\sum_i n_i \alpha_i\right)\left(\sum_j m_j \beta_j\right) = \sum_{i,j} n_i m_j \alpha_i \beta_j$$

where n_i , m_j are integers $\alpha_i \in [A , B]$, $\beta_j \in [B , C]$

(ii) For $[A , B]_{\overline{C}}$ = Quotient of the free abelian group in (i) modulo the subgroup generated by 0_{AB} (the null map) in case (ii) $\overline{C}$ contains C as a subcategory, has the same set of null morphisms as in C.

(iii) Free abelian group generated only by the distributive elements (the null elements are such) of hom sets $[A,B], [A,C]$ etc. Composition law (M) will be satisfied, since the composition of two distributive elements is again distributive.

§ 3. Some basic properties

Proposition 3.1. *Let* $(A_i)_{i \in I}$ (I finite) *be a collection of objects of* $\mathcal{A}$, *then the family* $\{A \xrightarrow{p_i} A_i\}$ *is a product for the family* $(A_i)_{i \in I}$, *if there exists uniquely determined monomorphisms* $u_j : A_j \to A$, *such that*

$$u_i \, p_j = \delta_{ij} \quad \text{and} \quad \Sigma p_k \, u_k = 1_A .$$

First part is essentially § 14.1 [4]. For the second part we use axiom i) to deduce

$$(\Sigma p_k \, u_k) \, p_j = p_j \quad \text{i.e.} \quad \Sigma p_k \, u_k = 1_A .$$

We notice that the "only if" part of Proposition 3.1, which holds in an additive category will no longer be true; since we fail to restore the uniqueness of the morphism into a direct product determined by the components as can be seen in the example below.

Consider the left additive category K as in Example 3. In this category the direct product of two objects G_1, G_2 coincides with the usual direct product $G = G_1 \times G_2$ of groups with $\pi_i : G \to G_i$ and $\sigma_i : G_i \to G$ $(i = 1, 2)$ as the group homomorphisms as projections and injections, as such they are distributive morphisms of K and satisfies the equation

$$\pi_1 \, \sigma_1 + \pi_2 \, \sigma_2 = 1 \quad \text{and} \quad \sigma_i \, \pi_j = \delta_{ij} .$$

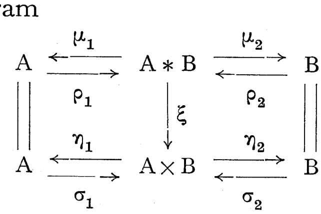
But G is not a coproduct of G_1, G_2; since the non trivial map $\alpha : G \to G$ of K defined by

$$\alpha(g_1, g_2) = \begin{cases} (g_1, g_2) & \text{when } g_1 \text{ and } g_2 \neq 0 \\ 0 & \text{when } g_1 \text{ or } g_2 = 0 \end{cases}$$

has trivial components. Thus in K^{op}, which is right additive G_i with $\sigma_i^{op} : G \to G_i$ and $\pi_i^{op} : G_i \to G$ satisfy the condition of Proposition 3.1, but fails to be a product.

Proposition 3.2. *If* $\mathcal{A}$ *is a right additive category with finite products and coproducts, then the canonical* $\xi : A * B \to A \times B$ *for two objects* A *and* B *of* $\mathcal{A}$ *is a retraction.*

Proof. In the diagram



the canonical morphism ξ is the unique map determined by the pairs $\sigma_1 : A \to A \times B$, $\sigma_2 : B \to A \times B$ or by the pairs $\mu_1 : A * B \to A$ and $\mu_2 : A * B \to B$. Thus

$$\xi = \langle \sigma_1 , \sigma_2 \rangle = \{ \mu_1 , \mu_2 \}$$

where the symbols $\langle \, , \rangle$ and $\{ \, , \}$ are used for the unique map from the coproduct and into a product determined by the components.

Now set

$$\eta = \pi_1 \rho_1 + \pi_2 \rho_2 : A \times B \to A * B.$$

Then

$$\begin{aligned} \eta\xi\pi_1 &= (\pi_1 \rho_1 + \pi_2 \rho_2)\, \xi\pi_1 \\ &= \pi_1 \rho_1 \xi\pi_1 + \pi_2 \rho_2 \xi\pi_1 \\ &= (\pi_1 \sigma_1 + \pi_2 \sigma_2)\, \pi_1 = 1_A \pi_1 = \pi_1 . \end{aligned}$$

Similarly $\eta\xi\pi_2 = \pi_2$, i.e. $\eta\xi = 1$.

COROLLARIES 1) *In a right additive category, every morphism* $f : A \times B \to C$ *is uniquely determined by the components* $\sigma_1 f$, $\sigma_2 f$.

2) *In a additive category every finite product is a biproduct.*

Now we shall state some *simple properties without proof.*

PROPOSITION 3.3. *If* $u : A \to B$ *is a morphism in a right additive category then the following conditions are equivalent*

(i) u *is a monomorphism*

(ii) $0 \to A$ *is the kernel of* u.

(iii) $1_A : A \to A$ *is the coimage of* u.

[coimage being defined as the cokernel of the kernel *whenever the required kernels cokernels exist*].

However for the dual

PROPOSITION 3.3*. *If* $\nu : A \to B$ *is a morphism, then the following two conditions are equivalent*

(i) $A \to 0$ *is the cokernel of* ν

(ii) $1_A : A \to A$ *is the image of* ν (image is the *dual* of the coimage).

Further for an epimorphism ν, *the equivalent conditions* (i) *and* (ii) *hold.*

We now assume that our category A has the further additional axiom.

Axiom I: Every morphism has a kernel and cokernel.

It is immediate that every morphism has then image and coimage.

From the definition of image and coimage, one can see by Fukawa's theorem [8]:

a) Every monomorphism is normal and every epimorphism is normal

b) *Every bimorphism is invertible.*

§ 4. RIGHT ABELIAN CATEGORIES

Let A *be a right additive category*, with axiom I. Then any morphism $\theta : A \to B$ in A admits a factorization

$$\begin{array}{ccc} A & \xrightarrow{\theta} & B \\ \downarrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle\mu} \\ \operatorname{coim}\theta & \xrightarrow[\bar{\theta}]{} & \operatorname{im}\theta \end{array}$$

where $\bar{\theta}$ is the unique map: coim $\theta \to$ im θ, making the diagram commutative.

DEFINITION 4.1. A right additive category with axiom I and with finite products and coproducts is called *right abelian* if the unique map $\bar{\theta}$ in the factorization of θ is an equivalence.

The dual of a *right abelian category is left abelian.*

PROPOSITION 4.2. *A right abelian category is a unique factorization category.*

Proof. Since every epimorphism is normal, every map θ admits a factorisation $\theta = \lambda\mu$ as above, unique to within equivalence [4].

PROPOSITION 4.3. *The necessary and sufficient condition that the sequence*

$$0 \to A \overset{\mu}{\to} B \overset{\nu}{\to} C$$

be exact is that the sequence

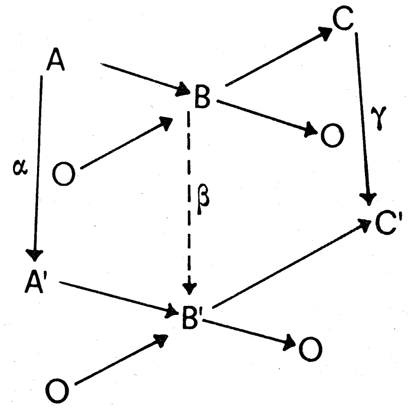
$$0 \to [X, A] \overset{\mu^*}{\to} [X, B] \overset{\nu^*}{\to} [X, C]$$

is exact in the category of Groups, for any object X. *Proof is easy.*

A version of five—lemma [11] for modules, now takes shape as

PROPOSITION 4.4.

Let



be a commutative diagram with exact diagonals, then there exists a unique $\beta : B \to B'$, *making the diagram commutative.*

(i) *If* γ *is a monomorphism so is* β.

(ii) *If* α *is an epimorphism so is* β.

(iii) *If* α *is an epimorphism and* γ *a monomorphism then* β *is an isomorphism.*

Proof: The existence of β is guaranteed by Fukawa's results (Corollary 2, § 3) and Proposition 9.

Second part follows because the composition of two normal epimorphisms is again normal.

Proof of usual nine lemma and further studies analogous to embedding theorems for abelian categories and distributively generated right additive categories i.e. in which hom sets [A , B] are based on a distributive set of generators S_{AB} in the sense of Fröhlich [2], will be left for the future; We notice however, that the proof of first and second isomorphism theorems and their duals for abelian categories, translates to right abelian categories, without change.

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