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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *Spectral Continuity and Permanent Sets in Topological Algebras.* Nota di EDWARD BECKENSTEIN, GEORGE BACHMAN e LAWRENCE NARICI, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — In questa nota X denota un'algebra topologica Hausdorff commutativa, completa, complessa, localmente *m*–convessa con unità.

Si ottengono condizioni sotto le quali le derivazioni di X proiettano X dentro il radicale di X. Inoltre si danno criteri secondo i quali X è un'algebra di Banach. È dimostrato che per Q–algebre imbarilate (« barreled ») X, se Ω è una collezione aperta nel piano complesso, allora l'estensione principale di Ω in X è aperta. È anche dimostrato che un Ω semplicemente connesso è permanente riguardo a X, così si generalizza il risultato di Ackermans (« On the Principal Extension of Complex Sets in a Banach Algebra », Indagationes Mathematicae, 1967, 146–150). Alla fine si dimostra che Q–algebre imbarilate X possiedono continuità spettrale e si presenta un esempio di un'algebra di Fréchet che non ha continuità spettrale.

Throughout this paper X and Y denote Hausdorff commutative complete locally *m*–convex complex topological algebras with identities denoted in each case by *e*. In certain cases we shall further hypothesize that X be a Q–algebra (i.e. that its set of units be open) or that X be barreled.

Through the use of a certain map ψ, some of whose properties are discussed in Sec. 1, some conditions are obtained (prior to Theorem 3) under which derivations of X map X into its radical. We also obtain some criteria under which X is a Banach algebra (Theorems 3 and 4). In Sec. 2 we show (Theorem 5) that for barreled Q–algebras X, if Ω is an open subset of the complex plane C, then its principal extension in X , M (Ω , X),

(*) Nella seduta del 20 febbraio 1971.

is open in X. We also show (Theorem 6) that if Ω is simply connected, then Ω is permanent with respect to X, thus generalizing the result of Ackermans [1, p. 147]. In Sec. 3 we show that barreled Q–algebras X have spectral continuity (Theorem 7) and then give an example of a Fréchet algebra which is not a Q–algebra and which fails to have spectral continuity.

1. The mapping ψ

Letting $\mathfrak{M}$ denote the set of closed maximal ideals of X, we identify $\mathfrak{M}$ and Φ, the collection of nontrivial continuous complex homomorphisms of X, and assume that $\mathfrak{M}$ carries the weakest topology which makes the maps $\hat{x} : \mathfrak{M} \to C$ taking $M \in \mathfrak{M}$ into $x(M)$ continuous for every $x \in X$ and $M \in \mathfrak{M}$. $C(\mathfrak{M})$ denotes the locally m–convex Hausdorff algebra of continuous maps of $\mathfrak{M}$ into C with compact-open topology and pointwise operations. Note that when $\mathfrak{M}$ is compact (as will be the case when X is a barreled Q–algebra), $C(\mathfrak{M})$ is a Banach algebra. Let ψ denote the map $x \to \hat{x}$ of X into $C(\mathfrak{M})$. We now investigate some of the properties of ψ. Among other things we prove that when $\psi(X)$ is closed in $C(\mathfrak{M})$, where X is a barreled Q–algebra, then derivations of X map X into its radical.

THEOREM 1. *If* X *is barreled, then* ψ *is continuous.*

Proof. A neighborhood base at 0 for the compact-open topology on $C(\mathfrak{M})$ is given by positive multiples of sets of the form $V = \{x \mid \sup |x(F)| \leq 1\}$ where F is a compact subset of $\mathfrak{M}$, so it suffices to show that the sets $\psi^{-1}(V)$ are neighborhoods of 0 in X. Any compact subset F of $\mathfrak{M}$ is $\sigma(X', X)$–bounded, so F is contained in the polar E^0 of some barrel E in X. By hypothesis E is a neighborhood of 0, so $F^0 = \psi^{-1}(V) \supset E^{00} \supset E$, and the proof is seen to be complete.

If X is a barreled Q–algebra, then $\mathfrak{M}$ is compact [4, p. 56]; hence the gauge $p_{\mathfrak{M}}$ of $\mathfrak{M}^0$ is a continuous seminorm. The spectral radius $r_\sigma(x)$ of $x \in X$ is defined to be $\sup_{M \in \mathfrak{M}} |x(M)|$ when this exists; otherwise we say that $r_\sigma(x) = \infty$. In C(R), the continuous real-valued functions on R, the reals, with compact-open topology, for example, all unbounded functions have infinite spectral radius since the closed maximal ideals of C(R) are in 1 — 1 correspondence with R.

THEOREM 2. *If* X *is a barreled* Q*–algebra, then* $r_\sigma(x) = p_{\mathfrak{M}}(x)$ *for every* $x \in X$.

Proof. When $r_\sigma(x) = 0$, the result is trivial. If $r_\sigma(x) > 0$, then $(x/r_\sigma(x)) \in \mathfrak{M}^0$ by the definition of $r_\sigma(x)$. Since $\mathfrak{M}$ is compact, then $\mathfrak{M}^0$ is a barrel and therefore $\mathfrak{M}^0 = \{x \mid p_{\mathfrak{M}}(x) \leq 1\}$. Consequently $p_{\mathfrak{M}}(x/r_\sigma(x)) \leq 1$ or $p_{\mathfrak{M}}(x) \leq r_\sigma(x)$. To see that the reverse inequality holds, suppose that $x \in a\mathfrak{M}^0$ where $a > 0$. Thus $|(x/a)(M)| \leq 1$ for every $M \in \mathfrak{M}$ and it follows that $r_\sigma(x) \leq a$, whence $r_\sigma(x) \leq p_{\mathfrak{M}}(x)$.

Suppose that X is a barreled Q–algebra so that $\mathfrak{M}$ is compact and $C(\mathfrak{M})$ is a Banach algebra with respect to sup norm. Consequently if $\hat{X} = \psi(X)$ is a closed subset of $C(\mathfrak{M})$, then $\hat{X}$ is a Banach algebra. It follows from Theorem 2 that X will be a closed subset of $C(\mathfrak{M})$ if and only if X is complete with respect to $p_{\mathfrak{M}}$. Since $C(\mathfrak{M})$ is semisimple, so is $\hat{X}$. Letting $D : X \to X$ be a derivation, it follows immediately that $\hat{D} : \hat{X} \to \hat{X}$, $\hat{x} \to (\psi D)\, x$ is a derivation of $\hat{X}$. Hence, according to the result of Johnson [3] for semisimple commutative complex Banach algebras with identity, $\hat{D}$ must be trivial. Thus, for every $x \in X$, $(\psi D) x = \hat{0}$ so, for any x and any $M \in \mathfrak{M}$, $(Dx)(M) = 0$ and it follows that Dx belongs to the radical of X. We summarize this as: If X is a barreled Q–algebra in which $\hat{X}$ is a closed subset of $C(\mathfrak{M})$, then any derivation of X maps X into its radical; consequently if X is semisimple, the only derivation of X is the trivial one. Our next theorem provides a case where X is a semisimple Banach algebra.

THEOREM 3. *If* X *is a barreled Q–algebra and the adjoint of* ψ, $\psi' : (\hat{X})' \to X'$ *is an onto map, then* X *is a semisimple Banach algebra.*

Proof. Since ψ is continuous by Theorem 1, it follows that ψ' is continuous and that $\psi'(\hat{X}') \subset X'$. Since, generally, $\ker \psi = \operatorname{Im}(\psi')^0$, in this case we have $\ker \psi = (X')^0 = \{0\}$ so that ψ must be 1 — 1. Thus, since $\ker \psi = \operatorname{Rad} X$, it follows that X is semisimple. Moreover since $\psi'(\hat{X}') = X'$, it follows [2, p. 517, Prop. 8.6.3] that as ψ is weakly continuous, therefore ψ^{-1} is weakly continuous. Thus ψ is an isomorphism and a homeomorphism (since $\hat{X}$ is bornological). Since X is complete, the result follows.

Definition 1. Let C (S) denote the Banach algebra of continuous, complex-valued functions on the compact Hausdorff space S with pointwise operations and sup norm. A closed subalgebra W of C (S) which separates points and contains the identity is called a *uniform algebra.*

If X is a semisimple barreled Q–algebra, then r_σ is a norm and we denote the topology it determines by $\mathcal{O}_r$.

THEOREM 4. *Let* X *be a semisimple barreled Q–algebra with topology* $\mathcal{O}$ *and continuous dual* X'. *Then* X *is isometrically isomorphic to a uniform algebra if and only if* $\mathcal{O}_r$ *is a topology for the dual pair* (X , X').

Proof. First assume that $\mathcal{O}_r$ is a topology for the dual pair. Since X is barreled, then $\mathcal{O} = \tau(X, X')$ where $\tau(X, X')$ denotes the Mackey topology. Since X is first countable when it carries $\mathcal{O}_r$ and $\mathcal{O}_r$ is a topology of the dual pair, then $\mathcal{O}_r = \tau(X, X')$. Thus $\mathcal{O} = \mathcal{O}_r$ and clearly the map $\psi : X \to \hat{X} \subset C(\mathfrak{M})$ taking x into $\hat{x}$ is an isometry as well as an isomorphism as $r_\sigma(x) = \|\hat{x}\|$. Since X is complete, $\hat{X}$ is closed in $C(\mathfrak{M})$, and since $\hat{X}$ clearly contains the identity and separates points of $\mathfrak{M}$, it follows that X is isometrically isomorphic to a uniform algebra.

Conversely, suppose that X is a uniform algebra. We begin by proving that $\mathcal{O}_r \subset \tau(X, X')$ so that $\mathcal{O}_r$ is a topology of the dual pair if and only if $\sigma(X, X') \subset \mathcal{O}_r$. To see that $\mathcal{O}_r \subset \tau(X, X')$, let H denote the $\sigma(X', X)$–

closure of the balanced convex hull of $\mathfrak{M}$ in X'. According to [6, pp. 30–31], $r_\sigma(x) = \sup_{h \in H} |h(x)|$. Moreover, since $\mathfrak{M}$ is compact, $H = \mathfrak{M}^{00}$ is compact in X', and we see that $\mathcal{O}_r$ is the topology of uniform convergence on the balanced convex compact set $\mathfrak{M}^{00}$, which immediately implies the desired result that $\mathcal{O}_r \subset \tau(X, X')$. Clearly $\sigma(X, X') \subset \mathcal{O}_r$ if and only if for any $f \in X'$ there is an $\varepsilon > 0$ such that $\{x \mid r_\sigma(x) \leqq \varepsilon\} = \varepsilon\mathfrak{M}^0 \subset \{f\}^0$. To complete the proof of the converse, we show that this last condition holds for any uniform algebra $X \subset C(S)$.

If $f \in X'$ then denote a continuous linear extension of f to C (S) by $\hat{f}$. Associated with $\hat{f}$ there is a regular Borel measure μ on S such that $\hat{f}(x) = \int_S x(s)\, d\mu$ for any $x \in C(S)$ and $\|\mu\| < \infty$. Identifying S and the maximal ideals of C (S), let $S \cap X$ denote the set of maximal ideals $\{M \cap X \mid M \in S\}$, and observe that $S \cap X \subset \mathfrak{M}$. Thus $(1/\|\mu\|)\,\mathfrak{M}^0 \subset (1/\|\mu\|)(S \cap X)^0$. If $x \in (1/\|\mu\|)\,\mathfrak{M}^0$, then $|x(s)| \leqq 1/\|\mu\|$ for all $s \in S$. Thus $|f(x)| \leqq (1/\|\mu\|) \int_S d|\mu| = 1$ which implies that $1/\|\mu\|\, \mathfrak{M}^0 \subset \{f\}^0$ and completes the proof.

2. Principal Extensions

For any subset Ω of the complex plane, we recall that the principal extension of Ω in X, M (Ω, X), consists of those $x \in X$ for which the spectrum of x, $\sigma(x)$, is contained in Ω. In complex commutative Banach algebras with identity, M (Ω, X) is open if Ω is open [1, p. 147, Theorem 1.1]. We show that this is not generally true for complete locally m–convex commutative Hausdorff topological algebras with identity. First we consider a category of algebras where it is true.

THEOREM 5. *If* X *is a barreled* Q*–algebra and* Ω *is open, then* M (Ω, X) *is open.*

Proof. In barreled Q–algebras $\sigma(x)$ is compact for any x [4, p. 77]. Since Ω is open, if $x \in M(\Omega, X)$ so that $\sigma(x) \subset \Omega$, then the distance $d(\sigma(x), \partial\Omega)$, from $\sigma(x)$ to the boundary of Ω, $\partial\Omega$, must be some positive number a. Noting that (as in Theorem 2) $\mathfrak{M}^0$ is a neighborhood of 0 in X, choose ε such that $0 < \varepsilon < a$, and consider $y \in x + \varepsilon\mathfrak{M}^0$. To show M (Ω, X) to be open, we wish to show that $\sigma(y) \subset \Omega$ so that a neighborhood of x lies in M (Ω, X). A typical element of $\sigma(y)$ is $y(M)$ where $M \in \mathfrak{M}$ and for any such M

$$|x(M) - y(M)| \leq \varepsilon < a = d(\sigma(x), \partial\Omega) \leq d(x(M), \partial\Omega)$$

from which it follows that $y(M)$ belongs to the open set Ω, and completes the proof.

A case where Ω is open but M (Ω, X) is not is given next.

Example 1. Consider the commutative complete locally m–convex Hausdorff algebra with identity C (C) of continuous, complex-valued functions on C with compact-open topology. Note that for any $x \in C(C)$, $\sigma(x) = \{x(t) \mid t \in C\}$. We show that for any proper open subset Ω of C containing 0, $M(\Omega, X)$ is not open.

Since $\sigma(0) = \{0\}$, we note that $x = 0 \in M(\Omega, X)$. The seminorms p_n $(n = 1, 2, \cdots)$, where $p_n(x) = \sup_{|t| \leq n} |x(t)|$, generate the compact-open topology on C (C) so that the neighborhoods $aV_n = \{x \in C(C) \mid p_n(x) \leq a\}$ $(a > 0)$ form a neighborhood base at 0 and we show that each aV_n contains some x_n whose spectrum is not in Ω. To construct x_n choose $\mu \notin \Omega$ and define $x_n(t) = 0$ for $|t| \leq n$, $x_n(t) = (|t| - n)\mu$ for $|t| > n$. Thus $x_n \in aV_n$ while $x_n(n+1) = \mu \notin \Omega$ so $x_n \notin M(\Omega, X)$ and the example is complete.

We further note that since C (C) is first countable, it is a Fréchet algebra—hence barreled. It is not a Q–algebra, however, since it contains a maximal ideal which is not closed: namely the maximal ideal containing the ideal I of continuous functions with compact support cannot be closed since I is dense in C (C). Thus " Q–algebra " cannot be removed from the hypothesis of Theorem 5.

Definition 2. Let $P = (p_\mu)_{\mu \in M}$ be a saturated system of multiplicative seminorms which generate the topology on X. A superalgebra $Y \supset X$ is said to be P–*compatible with* X if there exists a saturated system $Q = (q_\lambda)_{\lambda \in L}$ of multiplicative seminorms generating the topology on Y such that the collection $(q_{\lambda|X})$ of their restrictions to X is P.

For each $\mu \in M$ let $N_\mu = \{x \in X \mid p_\mu(x) = 0\}$ and let X_μ denote the completion of the normed algebra X/N_μ; let π_μ denote the continuous homomorphism of X into X_μ, $x \to x + N_\mu$. Now suppose that Y is P–compatible with X, let $\hat{N}_\mu = \{y \in Y \mid \hat{p}_\mu(y) = 0\}$ where $\hat{p}_\mu$ denotes an extension of $p_\mu \in P$ to Y, and let Y_μ denote the completion of $Y/\hat{N}_\mu$. It is easy to show in this case that the map $x + N_\mu \to x + \hat{N}_\mu$ embeds the Banach algebra X_μ isomorphically and isometrically in Y_μ, and we shall freely make this identification in what follows.

Definition 3. With P as in the preceding definition, the element $x \in X$ is said to be a *topological divisor of zero* if, for some index μ, $\pi_\mu x$ is a topological divisor of zero in the Banach algebra X_μ. An element $x \in X$ is a *strong* topological divisor of zero if the map $y \to xy$ is *not* a topological isomorphism of X into itself.

As indicated in [4], the second condition implies the first, but not much is known about the converse. Some further facts about topological divisors of zero are presented here. In regard to the first such result, note that in Banach algebras W, letting U denote the (open) set of units of W, the boundary of U, ∂U, is contained in the set of topological divisors of zero. In algebras X, the situation is not as simple because, letting U denote the units of X, $\partial U \cap U$ is not generally empty.

LEMMA 1. (a) *If* $x \in \bar{U}$ *and* $x \notin U$, *then* x *is a topological divisor of zero.* (b) *If* $\mu \in \sigma(x) \cap \partial\sigma(x)$, *then* $x - \mu e$ *is a topological divisor of zero.*

Proof. For $x \in \bar{U}$, $x \notin U$, choose a net $(x_s)_{s \in S}$ of points from U which converges to x. Since $x \notin U$, $\pi_\mu x$ is not invertible in X_μ for some $\mu \in M$ ([4]). Since, for each $s \in S$, $\pi_\mu x_s$ is invertible in X_μ, then $\pi_\mu x$ belongs to the boundary of the set of units of X_μ and is therefore a topological divisor of zero in X_μ. It follows that x is a topological divisor of zero in X. The proof of (b) is similar.

With notation as in Definition 2:

LEMMA 2. *If* X *is a closed subalgebra (with identity) of* Y, *then* $\partial\sigma_X(x) \cap \sigma_X(x) \subset \partial\sigma_Y(x) \cap \sigma_Y(x)$.

Proof. By the preceding lemma, if $\mu \in \partial\sigma_X(x) \cap \sigma_X(x)$, then $x - \mu e$ is a topological divisor of zero in X which implies that it is a topological divisor of zero in Y as well. Thus there is some $m \in L$ such that $\pi_m(x - \mu e)$ is a topological divisor of zero in Y_m, so that $\mu \in \sigma_Y(x)$. If $\mu \notin \partial\sigma_Y(x)$, then μ must be an interior point of $\sigma_Y(x)$. Since, clearly, $\sigma_Y(x) \subset \sigma_X(x)$, then μ would be an interior point of $\sigma_X(x)$ which is contradictory, so the proof is seen to be complete.

Definition 4. With notation as in Definition 2, a subset $\Omega \subset C$ is *permanent with respect to* X if for any P–compatible algebra Y, $M(\Omega, Y) \cap X = M(\Omega, X)$.

Note that it is generally true that $M(\Omega, X) \subset M(\Omega, Y) \cap X$. Moreover, as a simple consequence of Lemma 2, we have:

COROLLARY 1. *If* X *is a Q–algebra, then any simply connected region is permanent with respect to* X.

Proof. We need only note that in Q–algebras the spectrum of each element is closed, apply Lemma 2, and use [1, Theorem 2.1].

The following theorem (originally proved for Banach algebras by Ackermans [1, p. 147, Theorem 2.1]) is stronger than Corollary 1.

THEOREM 6. *In the notation of Definition* 2, *if* Ω *is simply connected, then* Ω *is permanent with respect to* X.

Proof. We need only show that $M(\Omega, Y) \cap X \subset M(\Omega, X)$ and to this end, let $x \in M(\Omega, Y) \cap X$. Since $\sigma_Y(x) \subset \Omega$ and $\sigma_Y(x) = \cup_{m \in L} \sigma_{Y_m}(\pi_m x)$ ([4]), then $\sigma_{Y_m}(\pi_m x) \subset \Omega$ for every $m \in L$. Since $X_m \subset Y_m$ for all $m \in M$ and Ω is simply connected, then $M(\Omega, Y_m) \cap X_m = M(X_m, \Omega)$ according to [1, p. 147]. Thus $\sigma_{X_m}(\pi_m x) \subset \Omega$ for each $m \in M$. It follows that $x \in M(\Omega, X)$ and completes the proof.

3. Spectral Continuity

X has spectral continuity if for any $\varepsilon > 0$ there is a neighborhood V of 0 such that for all $y \in X$, if $z \in y + V$, then $\sigma(z) \subset \sigma(y) + S_\varepsilon(0)$ and $\sigma(y) \subset \sigma(z) + S_\varepsilon(0)$ where $S_\varepsilon(0) = \{\mu \in C \mid |\mu| < \varepsilon\}$.

THEOREM 7. *Barreled Q–algebras have spectral continuity.*

Proof. Letting X_1 denote the completion of $\psi(X)$ in the Banach algebra $C(\mathfrak{M})$, we have for any $x \in X$ the fact that

$$\sigma_X(x) = \sigma_{C(\mathfrak{M})}(\hat{x}) = \sigma_{\psi(X)}(\hat{x}) = \sigma_{X_1}(\hat{x}). \tag{1}$$

Since X_1 is a Banach algebra, then for any $\hat{x} \in \hat{X} = \psi(X)$ and any $\varepsilon > 0$, there is a neighborhood $\hat{V}_\varepsilon$ of 0 in X_1 such that for any $f \in \hat{x} + \hat{V}_\varepsilon$

$$\sigma_{X_1}(f) \subset \sigma_{X_1}(\hat{x}) + S_\varepsilon(0) \quad \text{and} \quad \sigma_{X_1}(\hat{x}) \subset \sigma_{X_1}(f) + S_\varepsilon(0) \tag{2}$$

by [5, p. 36, Theorem 1.6.17]. Since X is barreled, ψ is continuous by Theorem 1 so $\psi^{-1}(\hat{V}_\varepsilon) = V_\varepsilon$ is a neighborhood of 0 in X. Now for any $y \in x + V_\varepsilon$, using (1) and (2) together,

$$\sigma_X(y) = \sigma_{X_1}(\hat{y}) \subset \sigma_{X_1}(\hat{x}) + S_\varepsilon(0) = \sigma_X(x) + S_\varepsilon(0)$$

and similarly

$$\sigma_X(x) = \sigma_{X_1}(\hat{x}) \subset \sigma_{X_1}(\hat{y}) + S_\varepsilon(0) = \sigma_X(y) + S_\varepsilon(0)$$

which completes the proof.

It is easy to verify that the Fréchet algebra C(R), where R denotes the reals, fails to have spectral continuity.

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