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On a theorem of uniform distribution of sequences of g-adic integers and a notion of independence

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Aritmetica. — *On a theorem of uniform distribution of sequences of g–adic integers and a notion of independence.* Nota di JAU–SHYONG SHIUE, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra (n. 3) una proposizione, sugli argomenti indicati nel titolo, che generalizza risultati noti relativi a casi più semplici.

1. INTRODUCTION

The notion of uniform distribution of sequences of g–adic integers, where g stands for a fixed rational integers with $g \geq 2$, was introduced by Meijer [4]. It is well known that the notion of uniform distribution of sequences of p–adic integers and that of uniform distribution of a sequence of rational integers, introduced by Niven [5], may be regarded as special cases of the notion of uniform distribution of sequences of g–adic integers. On the other hand, the field of g–adic numbers is an extension of the field of p–adic numbers: for a complete discussion, we refer to Mahler [3], chapter 1 and 2.

In this paper we first review the part of the theory of uniform distribution of sequences of g–adic integers which is needed for our purpose (section 2). Then we define a notion of independence of sequences of g–adic integers. Finally, section 3, we prove our main theorem.

2. UNIFORM DISTRIBUTION

DEFINITION 1. *Let g be an integer ≥ 2 and let Q_g denote the field of g–adic numbers. Let $d \in Q_g$ and k be a nonnegative integer. Then we define the set $\mathfrak{A}_k(d)$ by $\mathfrak{A}_k(d) = \{x \mid x \in Q_g, \mid x - d \mid_g \leq g^{-k}\}$, where $\mid x - d \mid_g$ denotes the g–adic pseudo-valuation of $x - d$.*

DEFINITION 2. *Let Z_g be the ring of g–adic integers, g an integer ≥ 2. Let $\{x_n\}$ $(n = 1, 2, \cdots)$ be a sequence in Z_g and let k be a positive integer. Let $A(\mathfrak{A}_k(d), N)$ denote the number of points x_n satisfying $x_n \in \mathfrak{A}_k(d)$ $(1 \leq \leq n \leq N)$. If $\lim_{N\to\infty} \frac{1}{N} A(\mathfrak{A}_k(d), N) = g^{-k}$ for $d = 0, 1, 2, \cdots, g^k - 1$, then we say that the sequence $\{x_n\}$ is k–uniformly distributed in Z_g.*

DEFINITION 3. *Let k be a fixed integer. The function ψ_k mapping Q_g into Q, Q being the field of rational numbers, is defined by $\psi_k(a) = \sum_{i=-\infty}^{k-1} a_i g^i$, where $\sum_{i=-\infty}^{\infty} a_i g^i$ is the g–adic representation of $a \in Q_g$.*

(*) Nella seduta del 20 febbraio 1971.

THEOREM 1. *Let k be a positive integer. The sequence $\{x_n\}$ $(n=1, 2, \cdots)$ in Z_g is k–uniformly distributed in Z_g if and only if* $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \psi_k(x_n)/g^k} = 0$ *for $h = 1, 2, \cdots, g^k - 1$, where $\psi_k(x_n)$ is given by definition 3.*

Now let G be an ordered pair of integers, $G = (g_1, g_2)$, with $g_1, g_2 \geq 2$. We denote by Z_G the direct product of the rings Z_{g_i}, $i = 1, 2$, i. e. $Z_G = Z_{g_1} \times Z_{g_2}$.

Let K be an ordered pair of non-negative integers, $K = (k_1, k_2)$ and let $D \in Z_G$, $D = (d_1, d_2)$. Then we call the set $\mathfrak{U}_{k_1}(d_1) \times \mathfrak{U}_{k_2}(d_2)$ the K–*neighborhood* of D, and denote it by $\mathfrak{U}_k(D)$.

DEFINITION 4. *Let $\{x_n\}$ $(n = 1, 2, \cdots)$ be a sequence in Z_G. If for fixed $\mathfrak{U}_k(D)$ in Z_G one has*

$$\lim_{N\to\infty} \frac{1}{N} A(\mathfrak{U}_k(D), N) = \frac{1}{g_1^{k_1} g_2^{k_2}},$$

then the sequence $\{x_n\}$ $(n = 1, 2, \cdots)$ is called (k_1, k_2)–uniformly distributed in Z_G.

On applying the character theory ([2], p. 109) and Eckmann criterion [1] in the product group $Z_g \times Z_g$, we have the following result.

THEOREM 2. *Let k be a positive integer. Then the sequence $\{(x_n, y_n)\}$ from $Z_g \times Z_g$ is (k_1, k_2)–uniformly distributed in $Z_g \times Z_g$, if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i (h_1 \psi_{k_1}(x_n)/g^{k_1} + h_2 \psi_{k_2}(y_n)/g^{k_2})} = 0,$$

for all $(h_1, h_2) \neq (0, 0)$, $h_1 = 0, 1, 2, \cdots, g^{k_1} - 1$; $h_2 = 0, 1, 2, \cdots, g^{k_2} - 1$.

DEFINITION 5. *Let Z_g be a ring of g–adic integers. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in Z_g and let k_1, k_2 be positive integers. $\{x_n\}$ and $\{y_n\}$ are said to be independent if*

$$\lim_{N\to\infty} \frac{1}{N} |\{n : x_n \in \mathfrak{U}_{k_1}(d_1), y_n \in \mathfrak{U}_{k_2}(d_2)\}| = \lim_{N\to\infty} \frac{1}{N} |\{n : x_n \in \mathfrak{U}_{k_1}(d_1)\}| \cdot \lim_{N\to\infty} \frac{1}{N} |\{n : y_n \in \mathfrak{U}_{k_2}(d_2)\}|$$

for all k_1 and k_2, where $|A|$ denotes the cardinal number of the set A.

3. MAIN THEOREM

THEOREM 3. *The sequence $\{(x_n, y_n)\}$ from $Z_g \times Z_g$ is (k_1, k_2)–uniformly distributed if and only if*

(1) *$\{x_n\}$ and $\{y_n\}$ are k_1–uniformly distributed in Z_g and k_2–uniformly distributed in Z_g respectively,*

(2) *$\{x_n\}$ and $\{y_n\}$ are independent.*

Proof. Necessity.

Let $\{(x_n, y_n)\}$ be (k_1, k_2)–uniformly distributed in $Z_g \times Z_g$. Then we have the limit relation mentioned in theorem 2. Take $h_1 = 0$, $h_2 \neq 0$, then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \frac{1}{g^{k_2}} h_2 \psi_{k_2}(y_n)} = 0, \quad \text{for } h_2 = 1, 2, \cdots, g^{k_2} - 1,$$

which implies that $\{y_n\}$ is k_2–uniformly distributed in Z_g, because of theorem 1. Similarly we have also $\{x_n\}$ k_1–uniformly distributed in Z_g. Now

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n : (x_n, y_n) \in \mathfrak{A}_{k_1}(d_1) \times \mathfrak{A}_{k_2}(d_2), 1 \leq n \leq N\}\right|$$

$$= \lim_{N\to\infty} \frac{1}{N} \left|\{n : x_n \in \mathfrak{A}_{k_1}(d_1), y_n \in \mathfrak{A}_{k_2}(d_2), 1 \leq n \leq N\}\right| = \frac{1}{g^{k_1} g^{k_2}},$$

because $\{(x_n, y_n)\}$ is (k_1, k_2)–uniformly distributed. Moreover, on account of the uniformity of $\{x_n\}$ and $\{y_n\}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n : x_n \in \mathfrak{A}_{k_1}(d_1)\}\right| = \frac{1}{g^{k_1}},$$

and

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n : y_n \in \mathfrak{A}_{k_2}(d_2)\}\right| = \frac{1}{g^{k_2}}.$$

Thus

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n : x_n \in \mathfrak{A}_{k_1}(d_1), y_n \in \mathfrak{A}_{k_2}(d_2), 1 \leq n \leq N\}\right|$$

$$= \lim_{N\to\infty} \frac{1}{N} \left|\{n : x_n \in \mathfrak{A}_{k_1}(d_1), 1 \leq n \leq N\}\right|$$

$$\cdot \lim_{N\to\infty} \frac{1}{N} \left|\{n : y_n \in \mathfrak{A}_{k_2}(d_2), 1 \leq n \leq N\}\right|,$$

which implies that $\{x_n\}$ and $\{y_n\}$ are independent.

Sufficiency. By the independence of $\{x_n\}$ and $\{y_n\}$, we have the last mentioned limit relation. Now, by hypothesis, we have

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n : x_n \in \mathfrak{A}_{k_1}(d_1), 1 \leq n \leq N\}\right| = \frac{1}{g^{k_1}}, \quad \text{for } d_1 = 0, 1, 2, \cdots, g^{k_1} - 1,$$

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n : y_n \in \mathfrak{A}_{k_2}(d_2), 1 \leq n \leq N\}\right| = \frac{1}{g^{k_2}}, \quad \text{for } d_2 = 0, 1, 2, \cdots, g^{k_2} - 1.$$

Therefore we obtain

$$\lim_{N\to\infty} \frac{1}{N} \left|\{n : (x_n, y_n) \in \mathfrak{A}_{k_1}(d_1) \times \mathfrak{A}_{k_2}(d_2)\}\right| = \frac{1}{g^{k_1} g^{k_2}},$$

and so $\{(x_n, y_n)\}$ is (k_1, k_2)–uniformly distributed in $Z_g \times Z_g$.

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