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On the extension of the equivalence relations to the sets of subsets

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 20 febbraio 1971

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the extension of the equivalence relations to the sets of subsets.* Nota di SERGE VASILACH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota preliminare si stabiliscono alcune proposizioni relative all'estensione delle relazioni di equivalenza agli insiemi delle parti, in vista di future applicazioni ai limiti induttivi di famiglie di spazi dotati di misure o di probabilità.

1. EXTENSION OF THE CANONICAL MAPPINGS

Suppose that A is a set, $\mathfrak{P}(A)$ is the set of subsets of A, R is an equivalence relation on A, A/R is the quotient set, φ is the canonical mapping of A onto A/R, and $\hat{\varphi}$ is the *extension* of φ to the sets of subsets defined by:

$$\hat{\varphi}(X) = \varphi\langle X\rangle \in \mathfrak{P}(A/R) \quad , \quad \forall X \in \mathfrak{P}(A) .$$

The mapping φ being surjective, $\hat{\varphi}$ is also surjective (cfr. [1], chap. II, § 5, n. 1, Prop. 1).

2. EXTENSION OF THE EQUIVALENCE RELATIONS

Let $\hat{R}$ be the equivalence relation on $\mathfrak{P}(A)$ associated with the surjective mapping φ, defined as follows: (cfr. [1], chap. II, § 6, n. 3):

$$\hat{R}\{X , Y\} \Longleftrightarrow (X \in \mathfrak{P}(A) , Y \in \mathfrak{P}(A) \quad \text{and} \quad \hat{\varphi}(X) = \hat{\varphi}(Y)) .$$

Let $\hat{\psi}$ be the canonical mapping of $\mathfrak{P}(A)$ onto the quotient set $\mathfrak{P}(A)/\hat{R}$. We have:

$$(1) \qquad \hat{R}\{X , Y\} \Longleftrightarrow \{\hat{\psi}(X) = \hat{\psi}(Y)\} \Longleftrightarrow \{\hat{\varphi}(X) = \hat{\varphi}(Y)\} ,$$

for any $X \in \mathfrak{P}(A)$, $Y \in \mathfrak{P}(A)$.

(*) Nella seduta del 20 febbraio 1971.

But, (1) implies

$$\{\hat{\psi}(X) = \hat{\psi}(Y)\} \Rightarrow \{\hat{\varphi}(X) = \hat{\varphi}(Y)\},$$

and since $\hat{\psi}$ is surjective, there exists (cfr. [1], chap. II, § 3, n. 9, prop. 9) a mapping $\hat{g}$ (uniquely determined by $\hat{\varphi}$) of $\mathfrak{P}(A)/\hat{R}$ into $\mathfrak{P}(A/R)$ such that

$$\hat{g} \circ \hat{\psi} = \hat{\varphi}. \tag{2}$$

If $\hat{s}$ is a section of $\hat{\psi}$, we have $\hat{g} = \hat{\varphi} \circ \hat{s}$. Relation (1) also implies:

$$\{\hat{\varphi}(X) = \hat{\varphi}(Y)\} \Rightarrow \{\hat{\psi}(X) = \hat{\psi}(Y)\}$$

with $\hat{\varphi}$ a surjective mapping. Hence (cfr. [1], *loc. cit.*) there exists a mapping $\hat{h}$ (uniquely determined by $\hat{\psi}$) of $\mathfrak{P}(A/R)$ into $\mathfrak{P}(A)/\hat{R}$ such that

$$\hat{\psi} = \hat{h} \circ \hat{\varphi}. \tag{3}$$

If $\hat{t}$ is a section of $\hat{\varphi}$, we have $\hat{h} = \hat{\psi} \circ \hat{t}$.

Therefore

$$\hat{\varphi} = \hat{g} \circ \hat{h} \circ \hat{\varphi} \qquad \text{and} \qquad \hat{\psi} = \hat{h} \circ \hat{g} \circ \hat{\psi};$$

hence

$$\hat{g} \circ \hat{h} = I_{\mathfrak{P}(A/R)} \qquad \text{and} \qquad \hat{h} \circ \hat{g} = I_{\mathfrak{P}(A)/\hat{R}}$$

where $I_{\mathfrak{P}(A/R)}$ (resp. $I_{\mathfrak{P}(A)/\hat{R}}$) is the identity mapping of $\mathfrak{P}(A/R)$ (resp. $\mathfrak{P}(A)/\hat{R}$).

Thus (cfr. [1], chap. II, § 3, n. 8, cor. de la Prop. 8) $\hat{g}$ and $\hat{h}$ are bijective mappings.

DEFINITION 1. *We say that* $\hat{R}$ *is the extension of* R *to the set of subsets of* A.

Thus we have

THEOREM 1. *Suppose* A *is a set,* R *is an equivalence relation on* A, φ *is the canonical mapping of* A *onto* A/R, $\hat{\varphi}$ *is the extension of* φ *to the set of subsets of* A, $\hat{R}$ *is the equivalence relation on* $\mathfrak{P}(A)$ *associated with* $\hat{\varphi}$, *and* $\hat{\psi}$ *is the canonical mapping of* $\mathfrak{P}(A)$ *onto* $\mathfrak{P}(A)/\hat{R}$. *Under these conditions, there exists a canonical bijection* (uniquely determined by $\hat{\psi}$ (resp. $\hat{\varphi}$)) of $\mathfrak{P}(A/R)$ *onto* $\mathfrak{P}(A)/\hat{R}$, and we have $\mathfrak{P}(A/R) = \mathfrak{P}(A)/\hat{R}$ (up to a *canonical bijection*).

Remark. Let Φ be the graph of φ; then $\Phi^{-1} \circ \Phi$ is the graph of R, $\hat{\Phi}$ is the graph of $\hat{\varphi}$ (cfr. [1], chap. II, § 5, n. 1) and $\hat{\Phi}^{-1} \circ \hat{\Phi}$ is the graph of $\hat{R}$ (cfr. [1], *loc. cit.* § 6, n. 2).

Example. Let G be an abelian multiplicative group, let H be a subgroup of G, φ the canonical mapping of G onto G/H; we have:

$$\{X\hat{R}Y \Longleftrightarrow \varphi\langle X\rangle = \varphi\langle Y\rangle \Longleftrightarrow X \cdot H = Y \cdot H\};$$

hence

$$\{X \equiv Y\,(\mathrm{mod} \cdot \hat{R})\} \Longleftrightarrow \{X \cdot H = Y \cdot H\},$$

$$\hat{\psi}(X) = \hat{g}^{-1}(\varphi\langle X\rangle) = \hat{g}^{-1}(X \cdot H)$$

and

$$X \cdot H = \hat{g}(\hat{\psi}(X)) \quad , \quad \forall X \in \mathfrak{P}(G).$$

3. THE EXTENSION OF MAPPINGS COMPATIBLE WITH EQUIVALENCE RELATIONS

PROPOSITION 1. *Let f be a mapping of a set* A *into a set* B, *compatible* (1) *with an equivalence relation* R. *Then the extension $\hat{f}$ is compatible with the equivalence relation* $\hat{R}$ *on* $\mathfrak{P}(A)$.

Proof. Let φ be the canonical mapping of A onto A/R; then (cfr. [1], *loc. cit.* § 6, n. 5, C 57) we have:

$$\{f \text{ compatible with R}\} \Longleftrightarrow$$

$$f = u \circ \varphi\,, \tag{4}$$

where u is a mapping of A/R into B, uniquely determined by f. If s is a section of φ, we have $u = f \circ s$, and (4) implies $\hat{f} = \hat{u} \circ \hat{\varphi}$.

But according to (2), we have:

$$\hat{f} = \hat{u} \circ \hat{g} \circ \hat{\psi}\,.$$

Let us set $\hat{w} = \hat{u} \circ \hat{g}$; $\hat{w}$ is a mapping (uniquely determined) of $\mathfrak{P}(A)/\hat{R}$ into $\mathfrak{P}(B)$. Such that $\hat{f} = \hat{w} \circ \hat{\psi}$, and this proves the proposition.

THEOREM 2. *Suppose that f is a mapping of the set* A *into the set* B, R *is an equivalence relation on* A, S *is an equivalence on* B, φ *is the canonical mapping of* A *onto* A/R, *and* θ *is the canonical mapping of* B *onto* B/S.

If f is compatible (2) *with the equivalence relation* R *and* S, *then the extension f is compatible with the equivalence relations* $\hat{R}$ *and* $\hat{S}$.

Proof. Let g be the mapping *induced* by f on passing to the quotients with respect to R and S; g is characterized (cfr. [1], *loc. cit.*) by the relation $\theta \circ f = g \circ \varphi$. (Cfr. fig. 1), and so

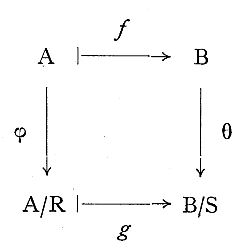

Fig. 1.

$$\hat{\theta} \circ \hat{f} = \hat{g} \circ \hat{\varphi}\,. \tag{5}$$

(1) For information concerning mappings compatible with an equivalence relations, cfr. [1], chap. II, § 6, n. 5.

(2) We say (cfr. [1], chap. II, § 6, n. 5) that f is compatible with the equivalence relations R and S, if $\theta \circ f$ is compatible with R; this means that

$$\{x \equiv x' \text{ (mod. R)}\} \Rightarrow \{f(x) = f(x') \text{ (mod}\cdot\text{S)}\}$$

If $\hat{\psi}$ (resp. $\hat{w}$) is the canonical mapping of $\mathfrak{P}(A)$ onto $\mathfrak{P}(A)/\hat{R}$ (resp. of $\mathfrak{P}(B)$ onto $\mathfrak{P}(B)/\hat{S}$), we have

$$\hat{\psi} = \hat{u} \circ \hat{\varphi} \qquad (\text{resp. } \hat{w} = \hat{v} \circ \hat{\theta}) \tag{6}$$

where $\hat{u}$ (resp. $\hat{v}$) is a well determined bijection (cfr. formula (3), n. 2).

Let us now consider the diagram shown in fig. 2, and let us set

$$\hat{\omega} = \hat{v} \circ \hat{g}\ , \tag{7}$$

where $\hat{\omega}$ is a mapping (uniquely determined) of $\mathfrak{P}(A/R)$ into $\mathfrak{P}(B)/\hat{S}$

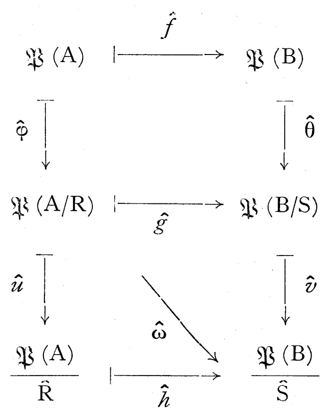

Fig. 2.

Let us consider, on the other hand, the diagram in fig. 3.

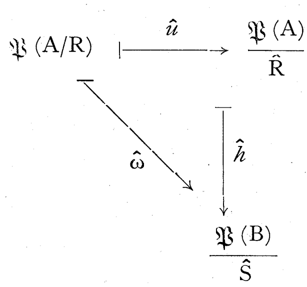

Fig. 3.

since $\hat{u}$ is bijective, we have:

$$\{\hat{u}(X) = \hat{u}(Y)\} \Rightarrow \{X = Y\} \Rightarrow \hat{\omega}(X) = \hat{\omega}(Y)\,.$$

Therefore (cfr. [1], chap. II, § 3, n. 9, Prop. 9) there exists a mapping h (uniquely determined by $\hat{\omega}$) of $\mathfrak{P}(A)/\hat{R}$ into $\mathfrak{P}(B)/\hat{S}$ such that

$$\hat{\omega} = \hat{h} \circ \hat{u}\,. \tag{8}$$

Then (7) and (8) yield

$$\hat{h} \circ \hat{u} = \hat{v} \circ \hat{g}\ . \tag{9}$$

On the other hand, using (9), we obtain

$$\{\hat{\theta}\circ\hat{f}=\hat{g}\circ\hat{\varphi}\}\Rightarrow\{\hat{v}\circ\hat{\theta}\circ\hat{f}=\hat{v}\circ\hat{g}\circ\hat{\varphi}=\hat{h}\circ\hat{u}\circ\hat{\varphi}\}.$$

But $\hat{v}\circ\hat{\theta}=\hat{w}$ and $\hat{u}\circ\hat{\varphi}=\hat{\psi}$; hence

$$\hat{w}\circ\hat{f}=\hat{h}\circ\hat{\psi}.$$

Therefore, $\hat{f}$ is compatible with the equivalence relations $\hat{R}$ and $\hat{S}$.

PROPOSITION 2. *Suppose that* A *and* B *are sets,* f *is a mapping of* A *into* B, S *is an equivalence relation on* B, u *is the canonical mapping of* B *onto* B/S, *and* R *is the equivalence relation on* A *associated with* $u\circ f$ (3).

Suppose, on the other hand, that $\hat{f}$ *is the extension of* f, $\hat{u}$ *is the extension of* u, $\hat{S}$ *is the extension of* S, $\hat{\theta}$ *is the canonical mapping of* $\mathfrak{P}(B)$ *onto* $\mathfrak{P}(A)/\hat{S}$ *and* $\hat{R}$ *is the equivalence relation on* $\mathfrak{P}(A)$ *associated with* $\hat{\theta}\circ\hat{f}$.

Under these conditions, the following Propositions are equivalent:

a) $X\hat{R}Y$

b) $\forall x\in X$, $\exists y\in Y$ and $\forall y\in Y$, $\exists x\in X$ such that

$$f(x)\equiv f(y)\ (\mathrm{mod}\cdot S).$$

Proof.

a) $\Rightarrow$ *b*) we have:

$$a)\iff X\hat{R}Y\iff\hat{\theta}(\hat{f}(X))=\hat{\theta}(\hat{f}(Y))\iff\hat{f}(X)\equiv\hat{f}(Y)\ (\mathrm{mod}\cdot\hat{S})\iff$$

$$\iff u\langle f\langle X\rangle\rangle=u\langle f\langle Y\rangle\rangle\Rightarrow\forall x\in X,\ \exists y\in Y\ \text{and}\ \forall y\in Y,\ \exists x\in X\sim$$

$$f(x)\equiv f(y)\ (\mathrm{mod}\cdot S)\Rightarrow b).$$

Conversely,

$$b)\Rightarrow\{u\langle f\langle X\rangle\rangle=u\langle fY\rangle\rangle\}\Rightarrow$$

$$\Rightarrow\{\hat{u}(\hat{f}(X))=\hat{u}(\hat{f}(Y))\}\Rightarrow$$

$$\Rightarrow\{\hat{\theta}(\hat{f}(X))=\hat{\theta}(\hat{f}(Y))\}\Rightarrow$$

$$\Rightarrow\{\hat{f}(X)\equiv\hat{f}(Y)\ (\mathrm{mod}\cdot\hat{S})\}\Rightarrow$$

$$\Rightarrow\{X\equiv Y\ (\mathrm{mod}\cdot\hat{R})\}\Rightarrow a).$$

COROLLARY 1. *Suppose that* A′ *is a subset of* A, R *is an equivalence relation on* A, u *is the canonical mapping of* A *onto* A/R, j *is the canonical injection of* A′ *into* A, *and* $R_{A'}$ *is the equivalence relation induced by* R *on* A′.

Suppose moreover that $\hat{J}$ *is the extension of* j, $\hat{u}$ *is the extension of* u, $\hat{R}$ *is the equivalence relation associated with* $\hat{u}$, $\hat{\theta}$ *is the canonical mapping of* $\mathfrak{P}(A)$ *onto* $\mathfrak{P}(A)/\hat{R}$, *and* $\hat{R}_{\mathfrak{P}(A')}$ *is the equivalence relation induced by* $\hat{R}$ *on* $\mathfrak{P}(A')$.

Under these conditions, the two following assertions are equivalent:

a) $X\equiv Y\ (\mathrm{mod}\cdot\hat{R}_{\mathfrak{P}(A')})$

b) $X\equiv Y\ (\mathrm{mod}\cdot\hat{R})$, $X\in\mathfrak{P}(A')$, $Y\in\mathfrak{P}(A')$.

(3) R is called the *inverse image* of S under f (cfr. [1], chap. II, § 6, n. 5).

Proof. We have:

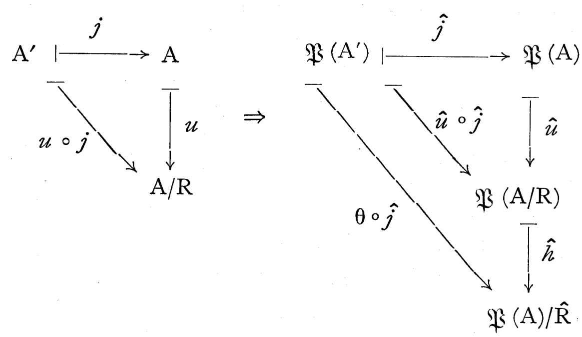

Fig. 4.

whence:

$$X \equiv Y \ (\mathrm{mod.}\ \hat{R}_{\mathfrak{P}(A')}) \Longleftrightarrow$$

$$\left\{ \begin{matrix} \hat{\theta}(\hat{j}(X)) = \hat{\theta}(\hat{j}(Y)) \\ X \in \mathfrak{P}(A'), Y \in \mathfrak{P}(A') \end{matrix} \right\} \Longleftrightarrow$$

$$\Longleftrightarrow \{\hat{h}(\hat{u}(\hat{j}(X))) = \hat{h}(\hat{u}(\hat{j}(Y)))\} \Longleftrightarrow \{\hat{u}(\hat{j}(X)) = \hat{u}(\hat{j}(Y))\} \Longleftrightarrow$$

$$\Longleftrightarrow \{\hat{u}(X) = \hat{u}(Y)\} \Longleftrightarrow \{u\langle X\rangle = u\langle Y\rangle\} \Longleftrightarrow$$

$$\Longleftrightarrow \{X \equiv Y \ (\mathrm{mod}\cdot \hat{R}), X \in \mathfrak{P}(A'), Y\, \mathfrak{P}\, \beta(A')\}.$$

COROLLARY 2. $\{\hat{j}$ *compatible with* $R_{A'}$ *and* $R\} \Rightarrow \{\hat{j}$ *compatible with* $\hat{R}_{\mathfrak{P}(A')}$ *and* $\hat{R}\}$.

Proof. Let us consider the diagram shown in fig. 5.

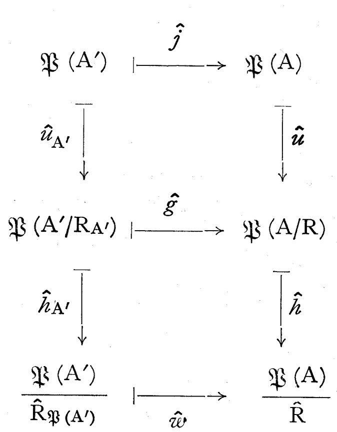

Fig. 5.

Let us prove that

$$\hat{u} \circ \hat{j} = \hat{g} \circ \hat{u}_{A'} \quad \text{implies}$$

$$\hat{w} \circ \hat{\theta}_{A'} = \hat{\theta} \circ \hat{j} .$$

To do this, we consider the mapping $\hat{h} \circ \hat{g}$ of $\mathfrak{P}(A'/R_{A'})$ into $\mathfrak{P}(A)/\hat{R}$, and the diagram in fig. 6.

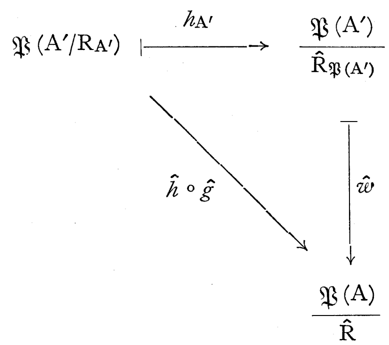

Fig. 6.

The mapping $\hat{h}_{A'}$ being a bijection, we have:

$$\{\hat{h}_{A'}(\hat{u}_{A'}(X)) = \hat{h}_{A'}(\hat{u}_{A'}(Y))\} \Rightarrow \{\hat{u}_{A'}(X) = \hat{u}_{A'}(Y)\} \Rightarrow$$

$$\Rightarrow \{(\hat{h} \circ \hat{g})(\hat{u}_{A'}(X)) = (\hat{h} \circ \hat{g})(\hat{u}_{A'}(Y))\} \Rightarrow$$

$$\Rightarrow \left\{ \text{there exists a mapping } \hat{w} \text{ of } \frac{\mathfrak{P}(A')}{\hat{R}_{\mathfrak{P}(A')}} \text{ into } \frac{\mathfrak{P}(A)}{\hat{R}} \text{ such that:} \right.$$

$$\left. \hat{h} \circ \hat{g} = \hat{w} \circ \hat{h}_{A'} \right\} .$$

Hence

$$\hat{h} \circ \hat{g} \circ \hat{u}_{A'} = \hat{w} \circ \hat{h}_{A'} \circ \hat{u}_{A'} .$$

But

$$\hat{g} \circ \hat{u}_{A'} = \hat{u} \circ \hat{j} \qquad \text{and} \qquad \hat{h}_{A'} \circ \hat{u}_{A'} = \hat{\theta}_{A} ,$$

whence

$$\hat{h} \circ \hat{u} \circ \hat{j} = \hat{w} \circ \hat{\theta} ;$$

and so

$$\hat{\theta} \circ \hat{j} = \hat{w} \circ \hat{\theta}_{A'} .$$

All results exposed in this paper will be utilised in our future studies on the direct limits of directed families of measure spaces and of probability spaces.

BIBLIOGRAPHIE

[1] BOURBAKI N., *Théorie des Ensembles*, chap. I–II, Paris, 19.