Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Vittorio Cantoni

On some plane solutions of the combined Einstein-Maxwell field equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.2, p. 156–162.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_50_2_156_0>

Fisica matematica. — *On some plane solutions of the combined Einstein–Maxwell field equations.* Nota di VITTORIO CANTONI (*), presentata (**) dal Corrisp. C. CATTANEO.

RIASSUNTO. — Si ricava una classe di soluzioni piane delle equazioni simultanee di Einstein e Maxwell, e se ne stabilisce l'equivalenza con una classe di soluzioni precedentemente ottenute da A. Perés [1], a meno di una trasformazione di coordinate che tuttavia generalmente altera la struttura differenziabile della varietà. Ne segue che in questa nuova forma la classe considerata viene a comprendere anche soluzioni per le quali il campo elettromagnetico soddisfa alle più deboli condizioni di regolarità fisicamente ammissibili (onde d'urto), senza che con ciò vengano violate le abituali condizioni di regolarità per il tensore metrico. Si considerano poi onde «sandwich» e si dimostra che il loro passaggio altera l'energia cinetica interna di una nuvola di particelle neutre di prova.

1. INTRODUCTION

From the point of view of general relativity, an electromagnetic wave propagating through flat space-time or, more generally, through a curved space-time with a prescribed metric, is an approximation which amounts to neglecting the electromagnetic energy-momentum tensor among the sources of the gravitational field. The set of simultaneous solutions of Maxwell's and Einstein's equations derived here can be regarded as providing an exact description of linearly polarized plane electromagnetic waves and of the gravitational field with which they are necessarily mixed on account of the presence of the electromagnetic energy-momentum tensor as a source term in the gravitational field equations. Whenever the electromagnetic tensor is assumed to be C^1, piecewise C^3, the solutions obtained in this paper are equivalent to a set of plane solutions first given by A. Perés [7]. However, the form of the solutions derived here allows a weakening of the regularity of the electromagnetic tensor, which can now be assumed to be only piecewise C^2, i.e. to satisfy the weakest physically admissible regularity conditions (see ref. [8], p. 15 and ref. [9], p. 356); the corresponding solutions can be interpreted as representing gravitational-electromagnetic shock waves. Next, "sandwich waves" are constructed and are shown to be able to modify the internal kinetic energy of a cloud of neutral test-particles.

(*) Lavoro eseguito nell'ambito dell'attività di ricerca del Gruppo Nazionale per la Fisica Matematica e per le Applicazioni della Matematica alla Fisica ed all'Ingegneria del Consiglio Nazionale delle Ricerche.

(**) Nella seduta del 20 febbraio 1971.

2. THE METRIC

Consider the "plane" (1) metric

$$(1)\qquad ds^2 = f\left(\frac{\xi^0 - \xi^1}{\sqrt{2}}\right)[(d\xi^2)^2 + (d\xi^3)^2] + (d\xi^1)^2 - (d\xi^0)^2$$

where, for the time being, the function f is only assumed to be positive, C^1 and piecewise C^3. Denoting by $\{\underset{i}{\varepsilon}\}$ (i and all latin indices $= 0, 1, 2, 3$ throughout) the natural basis associated with the coordinates $\{\xi^i\}$ at the generic point of space-time, it is convenient, for the physical interpretation of geometric quantities, to consider the field $\{\underset{i}{\boldsymbol{e}}\}$ of orthonormal tetrads defined by

$$(2)\qquad \underset{0}{\boldsymbol{e}} = \underset{0}{\varepsilon} \quad , \quad \underset{1}{\boldsymbol{e}} = \underset{1}{\varepsilon} \quad , \quad \underset{2}{\boldsymbol{e}} = f^{-1/2}\,\underset{2}{\varepsilon} \quad , \quad \underset{3}{\boldsymbol{e}} = f^{-1/2}\,\underset{3}{\varepsilon}\,.$$

On the other hand, calculations turn out to be simpler in the coordinate system $\{x^i\}$ defined by

$$x^0 \equiv u = \frac{1}{\sqrt{2}}(\xi^0 - \xi^1) \quad , \quad x^1 \equiv v = \frac{1}{\sqrt{2}}(\xi^0 + \xi^1),$$
$$x^2 \equiv y = \xi^2 \quad , \quad x^3 \equiv z = \xi^3$$

which will be used throughout this paper unless contrarily stated.

In these coordinates the metric (1) takes the form

$$(3)\qquad ds^2 = g_{ik}\, dx^i\, dx^k \equiv f(u)\,(dy^2 + dz^2) - 2\, du\, dv\,.$$

The only nonvanishing independent components of the Riemann and Ricci tensors (2) are, respectively,

$$(4)\qquad R_{2020} = R_{3030} = -\frac{1}{2}\frac{d^2 f}{du^2} + \frac{1}{4f}\left(\frac{df}{du}\right)^2 = -\sqrt{f}\,\frac{d^2\sqrt{f}}{du^2}$$

and

$$(5)\qquad R_{00} = \frac{1}{f}\frac{d^2 f}{du^2} - \frac{1}{2f^2}\left(\frac{df}{du}\right)^2 = \frac{2}{\sqrt{f}}\,\frac{d^2\sqrt{f}}{du^2}\,.$$

3. THE ELECTROMAGNETIC FIELD

The electromagnetic tensor F_{ik} is assumed to be such that, at each point, with respect to the tetrad defined by (2), the electric and magnetic vectors are mutually orthogonal and, more precisely, parallel to $\underset{2}{\boldsymbol{e}}$ and $\underset{3}{\boldsymbol{e}}$

(1) For definitions and discussions of the concept of "planeness" in curved space, see references (in particular ref. [6]).

(2) Apart for trivial changes in notation, Synge's conventions are used for the Riemann and Ricci tensors (ref. [7], pp. 15–17).

respectively, i.e.:

$$E_1 \equiv F_{ik} \underset{1}{e^i} \underset{0}{e^k} = 0 \quad , \quad E_2 \equiv F_{ik} \underset{2}{e^i} \underset{0}{e^k} = E(x^r),$$

$$E_3 \equiv F_{ik} \underset{3}{e^i} \underset{0}{e^k} = 0 \quad ,$$

$$H_1 \equiv F_{ik} \underset{2}{e^i} \underset{3}{e^k} = 0 \quad , \quad H_2 \equiv F_{ik} \underset{3}{e^i} \underset{1}{e^k} = 0,$$

$$H_3 \equiv F_{ik} \underset{1}{e^i} \underset{2}{e^k} = H(x^r),$$

where E and H are, at this stage, two undetermined functions of the coordinates. In the coordinates $\{x^i\}$ the above conditions are equivalent to the vanishing of all the components of F_{ik} except the following:

$$F_{20} = -F_{02} = \sqrt{f/2}\,(E+H)$$
$$F_{21} = -F_{12} = \sqrt{f/2}\,(E-H).$$

The further assumption that the electromagnetic field be purely radiative yields the single additional condition $E = H$, so that the resulting electromagnetic tensor has only two nonvanishing components:

$$(6) \qquad F_{20} = -F_{02} = \sqrt{2f}\,E(x^i) \quad ; \quad F_{ik} = 0 \quad \left(i, k \neq \begin{cases} 0,2 \\ 2,0 \end{cases}\right).$$

With these assumptions the electromagnetic energy-momentum tensor T_{ik} has only one non-vanishing component, namely

$$(7) \qquad T_{00} = 2\,E^2 ;$$

(notice that with respect to the tetrad $\{\underset{i}{e}\}$ the energy-momentum tensor has four nonvanishing components, namely

$$T_{ik} \underset{0}{e^i} \underset{0}{e^k} = T_{ik} \underset{1}{e^i} \underset{1}{e^k} = -T_{ik} \underset{0}{e^i} \underset{1}{e^k} = T_{ik} \underset{1}{e^i} \underset{0}{e^k} = E^2).$$

4. THE FIELD EQUATIONS

The metric (3) and the electromagnetic field (6) contain the functions $f(u)$ and $E(x^i)$ whose form must now be specified, to some extent, in order to satisfy Maxwell's and Einstein's field equations.

By leaving the function f still undetermined and writing Maxwell's equations explicitly, the latter are immediately seen to be satisfied, in the present case, if and only if $E(x^i)$ is a function of the coordinate $x^0 \equiv u$ only. Thus one can still dispose of the two undetermined functions $f(u)$ and $E(u)$ in order to satisfy Einstein's equations which reduce, on account of the vanishing of the curvature scalar and of all but one of the components of the Ricci and energy-momentum tensors, to the single equation

$$R_{00} = -\chi T_{00}$$

i.e., taking (5) and (7) into account,

$$\frac{d^2 \sqrt{f}}{du^2} = -\chi \sqrt{f}\, E^2. \tag{8}$$

Thus the function $f^{1/4}$ E can be prescribed arbitrarily, and the function f can then be determined by integration.

5. Comparison with the Peres solutions

Whenever the function $E(u)$ is C^1, piecewise C^3, the coordinate transformation

$$y' = y\sqrt{f} \quad , \quad z' = z\sqrt{f} \quad , \quad u' = u\,,$$

$$v' = v + \frac{1}{2}\sqrt{f}\,\frac{d\sqrt{f}}{du}\,(y^2 + z^2)\,,$$

is compatible with the differentiable structure of the manifold and transforms the metric (3) into the form

$$ds^2 = dy'^2 + dz'^2 - 2\,du'\,dv' + h(u')\,(y'^2 + z'^2)\,du'^2, \tag{3'}$$

$\left(h(u) \equiv \frac{1}{\sqrt{f}}\,\frac{d^2\sqrt{f}}{du^2} = -\chi E^2\right)$, which coincides with the plane solutions given by Perès. However, if the function $E(u)$ is *not* C^1, but is only piecewise C^2 (which is a physically admissible assumption as far as the electromagnetic tensor is concerned), the metric (3′) does not satisfy Lichnerowicz' regularity conditions while the metric (3) still does; (in the case the above coordinate transformation is not compatible with the differentiable structure of the manifold). In this sense the solutions of the form (3) are a generalization of the solutions of the form (3′), since they also include models of shock waves.

The coordinates associated with the metric (3) also turn out to be very convenient for the discussion of the effect of sandwich waves on neutral test-particles, as will be seen in section 9.

6. Monochromatic waves

Consider now the special case obtained by choosing the arbitrary function $f^{1/4}$ E of the form

$$f^{1/4}\,E = A\cos(ku)\,, \tag{9}$$

where A and k are constants. The corresponding solution will be referred to as *monochromatic*, and obviously reduces to an ordinary monochromatic linearly polarized plane wave in Minkowski space whenever the right hand side of equation (8) is neglected and one sets $f = 1$.

The solution of equation (8) is now

$$f = \left[a + bu - \frac{\chi A^2}{4\,k^2}\,(\sin^2 ku + k^2\,u^2)\right]^2 \tag{10}$$

where a and b are constants of integration.

7. Flat solutions

Whenever the electromagnetic field vanishes, $(\mathrm{d}^2 \sqrt{f}/\mathrm{d}u^2 = 0)$, our metric is flat, as can be seen from equation (4). In this case the solutions of equation (8) have the form

$$f = (a + bu)^2. \tag{11}$$

If moreover the constants a and b are chosen equal to 1 and 0, respectively, then $f = 1$ and the coordinates $\{\xi^i\}$ which have been used to write the metric (1) are cartesian. The possibility of choosing different values of the constants a and b will be useful for the construction of sandwich waves, since the coordinates $\{\xi^i\}$ cannot be simultaneously cartesian on both sides of such waves.

The flat metric

$$\mathrm{d}s^2 = \left[a + b\left(\frac{\xi^0 - \xi^1}{\sqrt{2}}\right)\right]^2 [(\mathrm{d}\xi^2)^2 + (\mathrm{d}\xi^3)^2] + (\mathrm{d}\xi^1)^2 - (\mathrm{d}\xi^0)^2, \tag{12}$$

corresponding to the solution (11) in the coordinates $\{\xi^i\}$, can be transformed into cartesian form by means of the transformation

$$\begin{aligned}
\xi^{0'} - \xi^{1'} &= \xi^0 - \xi^1 \\
\xi^{0'} + \xi^{1'} &= \xi^0 + \xi^1 + \frac{b}{\sqrt{2}}\,[(\xi^2)^2 + (\xi^3)^2]\left[a + b\left(\frac{\xi^0 - \xi^1}{\sqrt{2}}\right)\right] \\
\xi^{2'} &= \left[a + b\left(\frac{\xi^0 - \xi^1}{\sqrt{2}}\right)\right]\xi^2 \\
\xi^{3'} &= \left[a + b\left(\frac{\xi^0 - \xi^1}{\sqrt{2}}\right)\right]\xi^3 .
\end{aligned} \tag{13}$$

8. Sandwich waves

By "sandwich wave" is meant, here, a space-time which is flat everywhere except in a region lying between two lightlike hypersurfaces with equations $u = u_1$ and $u = u_2$ respectively (where u_1 and u_2 are constants). In the non-flat region an electromagnetic field is present, and Lichnerowicz' regularity conditions (namely, the continuity of the components of the metric tensor and of their first derivatives) are assumed to hold everywhere. (Compare with ref. (6) where purely gravitational sandwich waves are defined and discussed).

With the metric (3), the electromagnetic field (6) and the solutions (9) and (10) it is easy to fulfill the above requirements. An example is easily constructed by means of the following choice of the constants:

$u_1 = 0\,,\quad u_2 = n\pi \qquad$ (n positive integer);

$A = 0\,,\quad a = 1\,,\quad b = 0 \qquad$ in the region $u \leq u_1$;

$A = \bar{A} \neq 0\,,\quad \bar{A} < \frac{2}{u_2\sqrt{\chi}}\,,\quad a = 1\,,\quad b = 0 \qquad$ in the region $u_1 \leq u \leq u_2$;

$A = 0\,,\quad a = 1 + \frac{1}{4}\chi\bar{A}^2 u_2^2\,,\quad b = -\frac{1}{2}\chi\bar{A}^2 u_2 \qquad$ in the region $u \geq u_2$.

In this example the coordinates $\{\xi^i\}$ are cartesian in the region $u \leq u_1$, but not in the region $u \leq u_2$, where they can however be transformed into cartesian coordinates by means of the transformation (13).

In the particular model just constructed, the electromagnetic tensor is discontinuous at $u = u_1$ and $u = u_2$, so that the model should be regarded as representing an electromagnetic shock wave. However one could easily construct sandwich waves with a continuous electromagnetic field by choosing the arbitrary function $f^{1/4}\,E$ sufficiently smooth.

Whatever the form of the field in the non-flat region, it is easily seen that if the coordinates $\{\xi^i\}$ are required to be cartesian in the region $u \leq u_1$ they can no longer be such in the region $u \geq u_2$, or vice versa (except in the trivial case where the field vanishes). In fact, the condition $f = 1$ for $y \leq u_1$ and the regularity conditions on the metric imply $f = 1$ and $df/du = 0$ for $u = u_1$; on the other hand the function $d\sqrt{f}/du$ is decreasing, on account of (8), and vanishes at $u = 0$, so that $d\sqrt{f}/du$ is negative for $u > u_1$ and f is decreasing in the same region: therefore one cannot have $f = 1$ and $df/du = 0$ at $u = u_2$, and the coordinates $\{\xi^i\}$ cannot be cartesian in the region $u > u_2$.

9. The action of a sandwich wave on a cloud of neutral test-particles

Denote by F_1 and F_2 the flat regions $u \leq u_1$ and $u \geq u_2$ respectively, and by W the curved region $u_1 \leq u \leq u_2$. According to the last remark, the coordinates $\{\xi^i\}$, assumed to be cartesian in F_1, cannot be such in F_2 (and, of course, in W): however the lines $\xi^\alpha =$ constant ($\alpha = 1\,,2\,,3$), are immediately seen to be geodesics in all regions, and therefore represent the world-lines of neutral test-particles at rest, in the region F_1, with respect to the minkowskian frame associated with the coordinates. In order to examine the effect of the passage of the wave on a cloud of such particles one can apply, in the region F_2, the transformation (13) which immediately gives the parametric equations of the world-lines with respect to the minkowskian frame associated with the new coordinates $\{\xi^{i'}\}$: with respect to this frame only one of the particles is at rest (namely, the one with old space-coordinates

$\xi^\alpha = 0$), and the cloud is easily seen to be contracting ($b < 0$). Thus the passage of the wave has an effect which can be interpreted as a change in the internal kinetic energy of the cloud.

10. Remarks on the physical interpretation

It seems natural to interpret the waves considered in this paper as a mixture of electromagnetic and gravitational radiation, of the kind one would expect to find at large distances from a system of moving charges. This interpretation is supported by the remark that the gravitational aspect of the wave is essentially linked to the presence of the electromagnetic field, in the sense that whenever the latter vanishes the metric (1) can only be flat. The result of the last section expresses the fact that the wave, though of electromagnetic origin in the sense just explained, is able to interact with neutral matter (a result which has no analogue in special-relativistic electromagnetic theory and exhibits the gravitational aspect of the wave).

References

[1] N. Rosen, « Phys. Z. Sovjet », *12*, 366 (1937).
[2] A.H. Taub, « Ann. Math. », *53*, 472 (1951).
[3] G. C. Mc Vittie, « J. Rat. Mech. Anal. », *4*, 201 (1955).
[4] W. B. Bonnor, « Ann. Inst. Poincaré », *15*, 133 (1957).
[5] H. Bondi, « Nature », *179*, 1072 (1957).
[6] H. Bondi, F. A. E. Pirani e I. Robinson, « Proc. Roy Soc. » (London), *A 251*, 519 (1959).
[7] A. Peres, « Phys. Rev. », *118*, 1105 (1960).
[8] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme* (Masson, Paris 1955).
[9] J. L. Synge, *Relativity, the special theory* (North Holland publishing Co. Amsterdam 1958).
[10] J. L. Synge, *Relativity, the general theory* (North Holland publishing Co. Amsterdam 1960).