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A property of the uniform distributionon compact abelian groups with applications to characterization problems in probability

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Calcolo delle probabilità. — *A property of the uniform distribution on compact abelian groups with applications to characterization problems in probability.* Nota di PETER FLUSSER, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — È noto che se una variabile casuale ξ segue la distribuzione di Cauchy, allora per ogni numero reale γ la variabile casuale (*) $\eta = (\xi + \gamma)/(1 - \gamma\xi)$ segue anch'essa la distribuzione di Cauchy. Inversamente se per ogni numero reale γ, ξ e η sono identicamente distribuite come in (*) allora questa distribuzione è di Cauchy. William [7] ha provato che questo teorema è vero se la distribuzione di ξ e η è identica quando γ non è la tangente di un multiplo razionale di π.

In questa Nota noi proviamo che il risultato di William può derivare da certe proprietà delle cariabili casuali uniformemente distribuite rispetto a gruppi abeliani compatti. Per questa via si ottengono le caratterizzazioni di altre distribuzioni.

1. PRELIMINARIES

In this paper, groups will be compact second countable Abelian groups. We shall use additive notation, thus " + " will denote the group operation, " 0 " the identity, and $-g$ the inverse of the element g. If G is such a group, $g \in G$ and H is a subset of G, $g + H$ will denote the set $\{g + h : h \in H\}$. If n is an integer, ng will be the element $g + \cdots + g$ (n times) if $n > 0$, zero if $n = 0$, and $(-g) + \cdots + (-g)$ ($|n|$ times) if $n < 0$. We say that an element $g \in G$ generates the subgroup H of G if $H = \{ng : n = 0, \pm 1, \pm 2, \cdots\}$. Finally, B will denote the class of all Borel subsets of G; i.e. the σ-algebra generated by the open subsets of G.

We assume throughout that (Ω, S, P) is a probability space.

Definition 1. A random variable X (with values in G) is a function $X : \Omega \rightarrow G$ such that $X^{-1}(B) \in S$ for all $B \in B$. The *distribution* π of X is defined by:

$$\pi(B) = P\{\omega : X(\omega) \in B\} = P(X^{-1}(B)),$$

for all $B \in B$. This way (G, B, π) becomes a probability space. If X_1 and X_2 are two random variables we shall write $X_1 \sim X_2$ iff X_1 and X_2 have the same distribution.

Definition 2. The random variable X is said to be *uniformly distributed* on G iff π is the normalized Haar measure on G. [See [3] for the definition of Haar measure].

(*) Nella seduta del 20 febbraio 1971.

From the definition of Haar measure it follows that X is uniformly distributed on G iff $X \sim g + X$ for all $g \in G$.

We shall denote the group of real numbers in $[0, 1)$ under addition modulo $[0, 1)$ by $(T, \dot{+})$. We note that the normalized Haar measure on T is Lebesgue measure, m.

For a more detailed account of these ideas see [2] and [6].

2. A Characterization of the Uniform Distribution on G

Theorem 1. *Let* g_0 *generate a dense subgroup* H *of* G. *If* $X \sim g_0 + X$, *then* X *is uniformly distributed on* G.

We shall first prove the following:

Lemma: *Let* H *be a dense subset of* G. *If for all* $h \in H$, $X \sim h + X$, *then* X *is uniformly distributed on* G.

Proof: Let π be the distribution of X. For all E, $F \in B$, define:

$$\rho(E, F) = \pi(E \triangle F)$$

where $E \triangle F = (E \setminus F) \dot{\cup} (F \setminus E)$ is the symmetric difference of E and F, and the dot over the symbol $\cup$ indicates that the sets are disjoint.

Since $\pi(G) = 1 < \infty$, (B, ρ) becomes a complete metric space if sets whose symmetric difference is of π–measure zero are identified [[3], pp. 168 ff.].

Now let $g_0 \in G$ and $B_0 \in B$. The function $f: G \to B$ given by

$$f(g) = g + B_0$$

is continuous [[3], p. 268].

Let $\varepsilon > 0$. Since H is dense in G, we can find $h_0 \in H$ such that

$$\begin{aligned} \rho(f(g_0), f(h_0)) &= \rho(g_0 + B_0, h_0 + B_0) \\ &= \pi((g_0 + B_0) \triangle (h_0 + B_0)) < \varepsilon . \end{aligned}$$

We intend to show that:

$$\pi[(g_0+B_0) \cap (h_0+B_0)] \leq \pi(g_0+B_0) \leq \pi[(g_0+B_0) \cap (h_0+B_0)] + \varepsilon .$$

The first inequality is obvious since $(g_0+ B_0) \cap (h_0+B_0) \subset g_0 + B_0$, and the second follows from:

$$\begin{aligned} \pi(g_0 + B_0) &= \pi[\{(g_0+ B_0) \cap (h_0 + B_0)\} \ \dot{\cup} \ \{(g_0 + B_0) \setminus (h_0 + B_0)\}] \\ &= \pi[(g_0 + B_0) \cap (h_0 + B_0)] + \pi[(g_0 + B_0) \setminus (h_0 + B_0)] \\ &\leq \pi[(g_0 + B_0) \cap (h_0 + B_0)] + \pi[(g_0 + B_0) \triangle (h_0 + B_0)] \\ &\leq \pi[(g_0 + B_0) \cap (h_0 + B_0)] + \varepsilon \end{aligned}$$

Similarly we can show that

$$\pi[(g_0+B_0) \cap (h_0+B_0)] \leq \pi(h_0+B_0) \leq \pi[(g_0+B_0) \cap (h_0+B_0)] + \varepsilon .$$

Hence:

$$(*) \qquad |\pi(h_0 + B_0) - \pi(g_0 + B_0)| < \varepsilon .$$

Now since $X \sim h_0 + X$, $X \sim (-h_0) + X$. Hence:

$$P\{\omega : X(\omega) \in B_0\} = P\{\omega : (-h_0) + X(\omega) \in B_0\} = P\{\omega : X(\omega) \in h_0 + B_0\}.$$

Thus $\pi(B_0) = \pi(h_0 + B_0)$, and therefore, by (*)

$$|\pi(B_0) - \pi(g_0 + B_0)| < \varepsilon .$$

Since ε was chosen arbitrarily, $\pi(B_0) = \pi(g_0 + B_0)$. But g_0 and B_0 are arbitrary also, and since $\pi(G) = 1$, π is the normalized Haar measure on G. Hence X is uniformly distributed on G.

Proof of Theorem 1: Since $X \sim g_0 + X$, $X \sim (-g_0) + X$, and hence a simple induction argument yields: $X \sim ng_0 + X$ for every integer n. Thus $X \sim h + X$ for all $h \in H$, and since H is dense in G, the theorem is proved.

COROLLARY: *Let $\alpha \in T$ be an irrational number. Then the random* variable X with values in T is uniformly distributed iff $X \sim \alpha \dot{+} X$.

Proof: Since α is irrational, the cyclic group generated by α is dense in T; [[4], p. 34].

Remark: The requirement, in Theorem 1, that g_0 generate a *dense* subgroup of G is essential, as is illustrated by the following:

Example: Let $G = Z_4$, the cyclic group of order 4. Let the random variable X meet the values: 0 1 2 3

with probabilities: $\frac{1}{8}$ $\frac{3}{8}$ $\frac{1}{8}$ $\frac{3}{8}$.

Then $X \sim 2 + X$, but X is not uniformly distributed on Z_4.

3. APPLICATIONS TO REAL RANDOM VARIABLES

In this section we shall distinguish between the distribution of a real random variable as given in Definition 1, and its point distribution function. We shall, however, identify the real random variable uniformly distributed on [0, 1] with the random variable uniformly distributed on T.

THEOREM 2. *Let F be a strictly increasing continuous function on $[a, b)$, where $-\infty \leq a < b \leq +\infty$. Let the image of F be $[0, 1)$: i.e. $F(a) = 0$, $F(b - 0) = 1$. Let $\gamma \in [a, b)$ be such that $F(\gamma)$ is irrational.*

Let ξ be a real random variable, and define:

$$(1) \qquad \eta = F^{-1}((F(\xi) + F(\gamma)) \bmod [0, 1))$$

Then $\eta \sim \xi$ iff F is the distribution function of ξ.

Proof: If for all $\alpha, \beta \in [a, b)$ we define:

$$\alpha o \beta = F((F(\alpha) + F(\beta)) \bmod [0, 1))$$

then $([a, b), o)$ becomes a group isomorphic and homeomorphic to T provided the topology on $[a, b)$ is chosen to be the one induced on it by F^{-1} [[1], p. 253, ff.]. The theorem now follows from the corollary to Theorem 1 and that fact that F is the distribution of ξ iff $F(\xi)$ is uniformly distributed on T. [[5], p. 274, ff.].

Remark: We note that we have actually proved a stronger result than the converse of Theorem 2: If F is the distribution function of ξ, then for all $\gamma \in [a, b)$ $\xi \sim \eta$ where η is given by (1).

COROLLARY 1: [*A Characterization of the Uniform Continuous Distribution on* $[0, 1)$]

Let ξ be a real random variable, and for some irrational $\eta \in [0, 1)$ let

(C 1)
$$\eta = (\xi + \eta) \bmod [0, 1)$$

Then $\eta \sim \xi$ iff ξ is uniformly distributed on $[0, 1)$.

COROLLARY 2. [*A Characterization of the Cauchy Distribution*: *Williams* [7]].

Let ξ be a real random variable, and γ a real number not the tangent of a rational multiple of π. Let

(C 2)
$$\eta = \frac{\xi + \gamma}{1 - \gamma\xi} .$$

Then $\xi \sim \eta$ iff $\xi \sim Ca(0, 1)$, i.e. ξ has the distribution function

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \operatorname{arc\,tan} x , \qquad x \in R .$$

COROLLARY 3. [*A Characterization of the Exponential Distribution*].

Let ξ be a real random variable, and let γ be a positive real number not the logarithm of a rational number. Define:

(C 3)
$$\eta = \begin{cases} -\log(e^{-\xi} + e^{-\gamma} - 1) & \text{if } \xi \leq -\log(1 - e^{-\gamma}) \\ -\log(e^{-\xi} + e^{-\gamma}) & \text{if } \xi > -\log(1 - e^{-\gamma}) . \end{cases}$$

Then $\xi \sim \eta$ iff $\xi \sim \operatorname{Exp}(1)$, i.e. ξ has the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-x} & \text{if } x > 0 . \end{cases}$$

COROLLARY 4. [*A Characterization of the Hyperbolic Secant Distribution*].

Let ξ be a real random variable, and let γ be a real number, not the logarithm of the tangent of a rational multiple of π. For all non-zero x, let $L(x)$ be the odd function which is equal to $\log x$ for positive x. Define:

(C 4)
$$\eta = L\left(\frac{e^{\xi} + e^{\gamma}}{1 - e^{\gamma} e^{\xi}}\right) .$$

Then $\xi \sim \eta$ iff ξ is the absolutely continuous random variable with probability density function $f(x) \leq 1/\pi \operatorname{sech} x$, $x \in R$.

Proof of Corollaries 1–4.

In each case the distribution function of the random variable to be characterized satisfies the conditions of Theorem 2. An elementary computation then shows that substituting these functions for F in (1) of Theorem 2 yields (C 1)–(C 4) respectively. Thus for these choices of F, formula (1) of Theorem 2 is equivalent to formulas (C 1)–(C 4) respectively.

Thus in each case F is the distribution function of ξ iff $\xi \sim \eta$ and all the corollaries are proved.

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