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On polynomial mappings in linear spaces

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Topologia. — *On polynomial mappings in linear spaces* (*). Nota di ELISHA NETANYAHU e MEIR REICHAW, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Un'applicazione continua fra due spazi topologici uniformi è detta *polinomiale,* quando essa muti ogni successione generalizzata del primo spazio che sia priva di sottosuccessioni di Cauchy in una successione generalizzata del secondo spazio godente della stessa proprietà. Nel presente lavoro vengono assegnate condizioni sufficienti affinché un'applicazione continua risulti polinomiale, deducendone teoremi sull' iniettività di un'applicazione continua e varie applicazioni, fra cui un teorema di esistenza, unicità e dipendenza continua dai valori al contorno per un'equazione di Volterra di 2^a specie ed una generalizzazione del teorema fondamentale dell'algebra.

1. INTRODUCTION

The fundamental theorem of algebra can be expressed by saying that

(1) a polynomial $f \neq$ const. maps the complex plane X onto itself.

The following two properties are essential for (1) to hold:

(2) If $f(x_0) = y_0$ then the image $f(\mathrm{X})$ of the plane X covers some open (in X) neighborhood of y_0 (we say that f is open at (or in) y_0) and

(3) $f(x) \to \infty$ as $x \to \infty$.

Property (2) implies that the image $f(\mathrm{X})$ of X is an open subset of the plane X and property (3) (which, as known, characterizes polynomials in the class of all entire functions) implies that $f(\mathrm{X})$ is closed in X (see for example [16], p. 1399). Since the plane X is connected it follows that $f(\mathrm{X}) = \mathrm{X}$.

Because of the importance of (1) it is quite natural to generalize properties (2) and (3) to more general mappings and spaces than a polynomial and a plane. It is trivial that property (2) can be formulated without changes for mappings $f: \mathrm{X} \to \mathrm{X}$ of general topological spaces. Some results implying this property have been obtained i.a. in ([5], [6]) and [15]–[18]. An interesting theorem in this direction was recently obtained in [12]. Property (3) was generalized to the so-called polynomial mappings which play an import-

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ant role in finding conditions under which a mapping $f: X \to Y$ maps X onto Y (see for example [15] and [16]).

In section 3 of this paper the notion of a polynomial mapping is defined and some mappings $f: X \to Y$ of a linear space X into a linear space Y are shown to be polynomial mappings.

Section 4 contains a theorem generalizing a result obtained in [16]. It is applied to find i.a. conditions under which for a mapping $f: X \to Y$ the image $f(K)$ of a closed subset K of X is closed in Y. A simple theorem on homeomorphism of Banach spaces containing as a trivial consequence an existence theorem for the (not necessarily linear) Volterra equation of the second kind is proved. In Section 5 conditions are given under which for $f: X \to X$ the image $f(X)$ of X is dense in X and a fixed point theorem generalizing several recently obtained results about the contraction mapping principle is stated and applied i.a. to an implicit function theorem. Some theorems on " mappings onto " (surjective mappings) are also proved. As an application a generalization of the fundamental theorem of algebra is obtained. A remark concerning equations with quaternion coefficients is also given and two problems are posed.

2. Notation and definitions

We denote by $P(P')$ the set of pseudometrics $p(p')$ generating the uniformity of X (of Y), $\{x_\sigma\} = \{x_\sigma\}_{\sigma \in \Sigma}$ denotes a net (a generalized sequence) in X, i.e. $\{x_\sigma\}_{\sigma \in \Sigma}$ is a function mapping a directed set Σ into X and $\{x_{\sigma'}\} \subset \{x_\sigma\}$ means that $\{x_{\sigma'}\}$ is a subnet of $\{x_\sigma\}$. If X is a uniform space then the set of all Cauchy (fundamental) nets in X is denoted by $C(X)$ and the set of all nets $\{x_\sigma\}$ in X which *do not contain* a Cauchy subnet $\{x_{\sigma'}\}$ is denoted by $C'(X)$. The word " mapping " denotes a " continuous mapping ", " iff " stands for " if and only if " and " $=>$ " stands for " implies ". A subset $K \subset X$ of a pseudometric space X with pseudometric p is called p–bounded iff there exists $x_0 \in X$ such that $\sup_{x \in K} p(x_0, x) < \infty$.

Evidently the fact that K is p–bounded does not depend on x_0 (since if $\sup_{x \in K} p(x_0, x) < \infty$ then for each $y_0 \in X$ also $\sup_{x \in K} p(y_0, x) < \infty$). Given a set P of pseudometrics in X a subset $K \subset X$ is called P–bounded iff K is p–bounded for every $p \in P$. A net $\{x_\sigma\}$ in X is called almost p–bounded (ap–bounded) iff there exists $\sigma_0 \in \Sigma$ such that the set $\{x_\sigma ; x_\sigma \in \{x_\sigma\}$ and $\sigma \geq \sigma_0\}$ is p–bounded and a net $\{x_\sigma\}$ is aP–bounded iff $\{x_\sigma\}$ is ap–bounded for every $p \in P$.

Int U denotes the interior of the set U, $\| \quad \|$—the norm in a normed space and a linear space denotes a *linear Hausdorff space*. We note also that in the case that the topology of X is generated by a single pseudometric (or by a norm) the topological notions and the notion of completeness may be expressed, as well known, by using " usual " sequences $\{x_n\}_{n=1,2,\cdots}$.

3. POLYNOMIAL MAPPINGS IN LINEAR SPACES

In this section the notion of a polynomial mapping is defined and some mappings in linear spaces are shown to be polynomial mappings.

To generalize (3) one notes that if $\{x_n\}_{n=1,2,\ldots}$ is a sequence of points x_n of the plane X (and more generally of a finite dimensional Banach space X) then $x_n \to \infty$ iff $\{x_n\}_{n=1,2,\ldots}$ does not contain a Cauchy (fundamental) subsequence.

Thus (3) can be expressed by saying that $\{f(x_n)\}_{n=1,2,\ldots} \in C'(x)$ provided that $\{x_n\}_{n=1,2,\ldots} \in C'(X)$.

This leads to the following

DEFINITION 1. *A mapping* $f: X \to Y$ *of a uniform space* X *into a uniform space* Y *is called a polynomial mapping iff for every net* $\{x_\sigma\}$ *in* X *from* $\{x_\sigma\} \in C'(X)$ *follows that* $\{f(x_\sigma)\} \in C'(Y)$.

An important class of mappings used in analysis is the class of mappings of the form $f(x) = x - F(x)$ where $F: X \to X$ is a completely continuous mapping (not necessarily linear) of a Banach space X into itself. A generalization of the notion of a completely continuous mapping to uniform spaces may be given as follows:

DEFINITION 2. *A mapping* $F: X \to Y$ *of a uniform space* X *into a uniform space* Y *is called completely continuous iff for every* aP*–bounded net* $\{x_\sigma\}$ *in* X *there exists a subnet* $\{x_{\sigma'}\} \subset \{x_\sigma\}$ *and a point* $y \in Y$ *such that* $F(x_{\sigma'}) \to y$ (P*–denotes the set of pseudometrics generating the uniformity of* X).

As easily seen a completely continuous mapping $F: X \to Y$ of an *infinite* dimensional Banach space X into a Banach space Y is not a polynomial mapping.

(This follows from the fact that a ball in an infinite dimensional Banach space is not compact). Also if $F: X \to X$ is completely continuous the mapping $f(x) = x - F(x)$ need not be a polynomial mapping.

In the following Theorems 1 and 2 conditions are given for a mapping $f(x) = x - F(x)$ where $F: X \to X$ is completely continuous to be a polynomial mapping.

THEOREM 1. *Let* X *be a complete linear (Hausdorff) space and let* $X_1 \subset X$. *Let* $F: X_1 \to X$ *be completely continuous and suppose that* $f(x) = x - F(x)$ *satisfies the following condition*

(4) *If* $\{x_{\sigma'}\}$ *is a net in* X_1 *which is not* aP*–bounded then* $\{f(x_{\sigma'})\}$ *is not* aP*–bounded* (1).

Then $f: X_1 \to X$ *is a polynomial mapping.*

Proof. Let $\{x_\sigma\} \in C'(X_1)$ and suppose to the contrary that for some subnet $\{x_{\sigma'}\} \subset \{x_\sigma\}$ one has $\{f(x_{\sigma'})\} \in C(X)$. Thus $\{f(x_{\sigma'})\}$ is aP–bounded and by (4), $\{x_{\sigma'}\}$ is aP–bounded. Since F is completely continuous there exists a subnet $\{x_{\sigma''}\} \subset \{x_{\sigma'}\}$ and a point $y \in X$ with $F(x_{\sigma''}) \to y$ and by $\{f(x_{\sigma'})\} \in C(X)$ and the completeness of X it follows that there exists $x \in X$ such that $f(x_{\sigma''}) \to x$. Hence $x_{\sigma''} \to x + y$ contradicting $\{x_\sigma\} \in C'(X_1)$.

(1) P is the set of pseudometrics generating the uniformity of the linear space X.

COROLLARY 1. *If* $F : X \to X$ *is a linear completely continuous mapping of a Banach space* X *into itself and if* $x - F(x) = 0$ *implies* $x = 0$ *then* $f(x) = x - F(x)$ *is a polynomial mapping.*

Proof. As easily seen there exists a constant $m > 0$ such that $\|x - F(x)\| \geq m \|x\|$. It remains to apply Theorem 1.

THEOREM 2. *Suppose that the pseudometrics* p *of the set* P *generating the uniformity of the linear space* X *satisfy*

$$p(x, y) = p(x - y, 0) \tag{5}$$

and let $F : X_1 \to X$ *be a completely continuous mapping of a subset* X_1 *of* X *into* X *such that*

$$F(X_1) \text{ is P–bounded.} \tag{6}$$

Then $f(x) = x - F(x)$ *is a polynomial mapping.*

Proof. Let $\{x_\sigma\} \in C'(X_1)$ and suppose to the contrary that for some subnet $\{x_{\sigma'}\} \subset \{x_\sigma\}$ one has $\{f(x_{\sigma'})\} \in C(X)$. Then $\{f(x_{\sigma'})\}$ is aP–bounded. Hence by (5) and (6), $\{x_{\sigma'}\}$ is aP–bounded. Since F is completely continuous there exists $y \in X$ such that $F(x_{\sigma''}) \to y$ for some subnet $\{x_{\sigma''}\} \subset \{x_{\sigma'}\}$. It follows by $\{f(x_{\sigma'})\} \in C(X)$ and by (5) that $\{x_{\sigma''}\} \in C(X_1)$ contradicting $\{x_\sigma\} \in C'(X_1)$.

4. THE MAIN PROPERTY OF POLYNOMIAL MAPPINGS

This section contains i.a. two theorems. The first one (Theorem 3) generalizes a result obtained in [16]. The proof of it is a slight change of that of Lemma 1 in [16], p. 1399 and the theorem itself gives the main property of polynomial mappings. The second one (Theorem 4) has as a trivial consequence an existence theorem for the (not necessarily linear) Volterra equation of the second kind. Several other applications of Theorem 3 are also given.

THEOREM 3. *If* $f : X \to Y$ *is a polynomial mapping of a complete uniform space* X *into a Hausdorff uniform space* Y *then for each closed subset* K *of* X *the set* $f(K)$ *is closed in* Y. *(In particular* $f(X)$ *is closed in* Y*).*

Proof. Let $\{y_\sigma\}$ be a net such that $y_\sigma \in f(K)$ and $y_\sigma \to y$. Let x_σ be points of K with $y_\sigma = f(x_\sigma)$. If there would be $\{x_\sigma\} \in C'(X)$ then, since f is a polynomial mapping, also $\{y_\sigma\} \in C'(Y)$ contradicting $y_\sigma \to y$. Hence, for some subnet $\{x_{\sigma'}\}$ of $\{x_\sigma\}$ one has $\{x_{\sigma'}\} \in C(X)$. Since X is complete there exists a point x such that $x_{\sigma'} \to x$ and since K is closed one has $x \in K$. But then $y = f(x) \in f(K)$.

We give now several applications.

APPLICATION 1. Let $F : X \to X$ be a completely continuous mapping of a complete linear (Hausdorff) space X into itself and suppose that $X = \bigcup_{n=1}^{\infty} X_n$ where X_n are closed in X and P–bounded. Let $f(x) = x - F(x)$.

Then $f(X)$ is an F_σ–set in X. In particular in case that X is a complete linear metric space, $F(X)$ is an F_σ–set in X.

Proof. For every $n = 1, 2, \cdots$ the mapping $f \mid X_n : X_n \to X$ (f restricted to X_n) satisfies the assumptions of Theorem 1 (with X_1 replaced by X_n). Thus $f : X_n \to X$ is a polynomial mapping. By Theorem 3 the sets $f(X_n)$ are closed.

Easy examples show that under assumptions made in the above application $f(X)$ need not be closed.

APPLICATION 2. If $F : X_1 \to X$ is a completely continuous mapping of a closed subset X_1 of a Banach space X into X such that $F(X_1)$ is bounded, then for $f(x) = x - F(x)$ the set $f(X_1)$ is closed in X. In particular (as well known) if the closure $\overline{F(X_1)}$ is compact, then $f(X_1)$ is closed.

Proof. Follows from Theorems 2 and 3.

The reasoning in Application 3 is a generalization of that used in the theory of partial differential equations in case $f = D$ is a differential operator (see for instance [11], p. 57).

APPLICATION 3. Let X and Y be two complete metric spaces with the metrics ρ and ρ' respectively. Let $G \subset X \times Y$ be a closed subset of $X \times Y$ with the metric $\tilde{d}((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho'(y_1, y_2)$. Suppose that G is a graph of a function $f : X_1 \to Y$ (not necessarily continuous), where $X_1 \subset X$ and define a "new" metric on X_1 by $d(x_1, x_2) = \rho(x_1, x_2) + \rho'(f(x_1), f(x_2))$. Then X_1 with the metric d is a complete space.

Proof. We note first that by Theorem 3, if $g : Z \to W$ is a mapping of a complete uniform space Z into a uniform Hausdorff space W with uniformities generated by P and P′ respectively such that for each $p \in P$, there exists $p' \in P'$ and a constant $\alpha = \alpha(p, p') > 0$ satisfying $p'(g(x_1), g(x_2)) \geq \alpha p(x_1, x_2)$ for all x_1, x_2 of Z, then $g(Z)$ is closed in W. Now, the projection $\Pi : G \to X$ is an isometry on G and since G is closed in $X \times Y$ and $X \times Y$ is complete, one gets by putting $g = \Pi$, $Z = G$ and $W = X$ that $\pi(G) = X_1$ is closed in X. Thus X_1 with the metric d is a complete space.

Before the next application we introduce (see [1] or [7]) the notion of asymptotically equal mappings.

DEFINITION 3. *A mapping $f : X \to Y$ of a Banach space X into a Banach space Y is said to be asymptotically equal to a mapping $g : X \to Y$ iff*

$$\frac{\|f(x) - g(x)\|}{\|x\|} \to 0 \qquad \text{for} \quad \|x\| \to \infty .$$

APPLICATION 4. If $f : X \to Y$ is a mapping of a *finite* dimensional Banach space X into a Banach Y which is asymptotically equal to a linear mapping $g : X \to Y$ such that $\inf_{\|x\|=1} \|g(x)\| > 0$, then $f(K)$ is closed in Y, for every closed subset $K \subset X$.

Proof. By Theorem 3 it suffices to show that $f : X \to Y$ is a polynomial mapping. Thus let $\{x_n\} \in C'(X)$ and suppose to the contrary that $f(x'_n) \to y_0 \in Y$ for some subsequence $\{x'_n\}$ of $\{x_n\}$. Then by $\|x'_n\| \to \infty$ and $\dfrac{f(x'_n) - g(x'_n)}{\|x'_n\|} \to 0$ it follows that $\dfrac{g(x'_n)}{\|x'_n\|} =$ $= g\left(\dfrac{x'_n}{\|x'_n\|}\right) \to 0$ as $n \to \infty$ contradicting $\inf_{\|x\|=1} \|g(x)\| > 0$.

Before proving Theorem 4 we introduce a property T which was stated in [16], p. 1401 for the case of generalized F–spaces.

PROPERTY T. Let $f: X \to X$ be a mapping of a linear normed space X into itself such that for the point $y_0 \in f(X)$ there exists a point $x_0 \in f^{-1}(y_0)$, a ball (a spherical region) $B(x_0, r)$, a complex function $\lambda = \lambda(x_0) \neq 0$ and a real function $\alpha = \alpha(x_0) : 0 < \alpha < 1$ such that for every two points x and y belonging to $B(x_0, r)$ one has

$$\|x - y - \lambda(x_0)(f(x) - f(y))\| < \alpha \|x - y\| \qquad \text{for} \quad x \neq y .$$

Then $f: X \to X$ is said to have property T at (or in) the point $y_0 \in f(X)$.

The following result was in fact proved in Lemmas 4 and 3 of [16] (see also Theorem 3 of [5], p. 192).

(7) If $f: X \to X$ has property T at the point $f(x_0) = y_0$ then

$f[B(x_0, r)]$ contains a sphere $B(y_0, r_0)$ with radius $r_0 = \frac{1-\alpha}{|\lambda|} r$. (Evidently f is one to one on $B(x_0, r)$ and as easily seen f is a local homeomorphism).

REMARK 1. Let us note that Theorem 1.11 in [10], p. 12 is a simple consequence of (7).

THEOREM 4. *If* $F: X \to X$ *is a mapping of a Banach space* X *into itself such that for all* x, y *of* X : $\|F(x) - F(y)\| \leq \gamma \|x - y\|$, *where* $\gamma < 1$ *does not depend on* x, y *then* $f: X \to X$ *defined by* $f(x) = x - F(x)$ *is a homeomorphism of* X *onto itself.*

Proof. One has:

$$\|f(x) - f(y)\| = \|x - y - F(x) + F(y)\| \geq \|x - y\| - \|F(x) - F(y)\| \geq (1 - \gamma)\|x - y\| .$$

Thus f is a one to one polynomial mapping. By Theorem 3, f maps closed sets (in X) onto closed sets. It remains to show that $f: X \to X$ is an open mapping of X onto itself. For this purpose it suffices by (7) and the connectedness of X to prove that $f: X \to X$ has property T at every point $y_0 \in f(X)$. But this is quite trivial. Indeed, fix an arbitrary $\lambda, 0 < \lambda < 1$ and put $\alpha = 1 - \lambda + \gamma\lambda$. Then $0 < \alpha < 1$ and $\|x - y - \lambda(f(x) - f(y))\| \leq \leq \alpha \|x - y\|$.

We give an application of Theorem 4.

APPLICATION 5. Take the Volterra equation of the second kind (not necessarily linear)

$$x(t) - \int_{t_0}^{t} K(t, s, x(s))\, ds = y(t) . \tag{8}$$

Suppose that $K(t, s, u)$ is defined for

$$t_0 \leq s \leq t \leq t_0 + h \qquad \text{and} \qquad -\infty < u < \infty$$

and that $F(x) = F(x)(t) = \int_{t_0}^{t} K(t, s, x(s))\, ds$ is a continuous function of t.

Suppose also that

$$|K(t, s, u) - K(t, s, v)| < L|u - v| \qquad \text{for all} \quad u \neq v$$

$-\infty < u, v < \infty$ where L is a constant which does not depend on u, v (i.e., $K(t, s, u)$ satisfies the Lipschitz condition with respect to u). Then for h with $0 < h < \frac{1}{L}$ and $\gamma = Lh$ one has $\|F(x) - F(y)\| < \gamma \|x - y\|$, where $F: X \to X$ maps the Banach space $X = C[t_0, t_0 + h]$ of all continuous real functions $x = x(t), t_0 \leq t \leq t_0 + h$ with norm $\|x\| = \sup_{t_0 \leq t \leq t_0+h} |x(t)|$ into itself. By Theorem 4, equation (8) has for every $y = y(t)$ of $X = C[t_0, t_0 + h]$ a unique solution $x = x(t) \in X$, which depends continuously on $y = y(t)$.

REMARK 2. Let us note that a similar result can be stated for the equation

$$x(t) - \lambda \int_a^b K(t, s, x(s))\, ds = y(t)$$

with $K(t, s, u)$ defined for $a \leq s \leq t \leq b, -\infty < u < \infty$ and $|\lambda|$ sufficiently small.

Let us also note that taking in (8) $K(t, s, u) = f(s, u)$ and $y(t) \equiv x_0$ one obtains as known (under the assumptions made in the above Application) the existence, uniqueness and continuous dependence on x_0 of a solution $x(t), t_0 \leq t \leq h$ of the equation $x(t) - \int_{t_0}^t f(s, x(s))\, ds = x_0$.

5. A FIXED POINT THEOREM AND MAPPINGS ONTO

In this section conditions are given under which for $f: X \to X$ the image $f(X)$ of X is dense in X and a fixed point theorem generalizing some recent results about the contraction mapping principle, obtained by a number of writers, is stated and applied i.a. to an implicit function theorem. Some applications to "mappings onto" are obtained. A generalization of the fundamental theorem of algebra is proved. A remark concerning equations with quaternion coefficients is given and two problems are posed.

The idea of the next lemma is given in [8].

LEMMA 1. *Let $f: X \to X$ be a mapping of a metric space X with metric ρ into itself and let $x_0 \in X$. Denote $\underbrace{f \cdots f}_{s}(x_0) = f^s(x_0)$ and suppose that:*

(9) for every $\varepsilon > 0$ there exist $l \neq k$ such that $\rho(f^k(x_0), f^l(x_0)) < \varepsilon$

and

(10) $\rho(x_0, f^s(x_0)) \leq \rho(f(x_0), f^{s+1}(x_0))$ for $s = 1, 2, \cdots$.

Denote by $R = R(x_0)$ the set $\{x_0, f(x_0), ff(x_0), \cdots\}$.

Then $x_0 \in \overline{f(R)}$ (the closure of $f(R)$).

Proof. Take $\varepsilon > 0$ and let $l > k$ satisfy (9). Then by (10) and (9)

$$\rho(x_0, f(f^{l-k-1}(x_0))) = \rho(x_0, f^{l-k}(x_0)) \leq \rho(f^k(x_0), f^l(x_0)) < \varepsilon.$$

THEOREM 5. *If* $f: \mathrm{X} \to \mathrm{X}$ *is a mapping of a complete metric space* X *into itself such that* (9) *holds for every* $x_0 \in \mathrm{X}$ *and if*

$$\rho(f(x), f(y)) \geq \rho(x, y) \qquad \text{for all} \quad x, y \in \mathrm{X} \tag{11}$$

then $f(\mathrm{X}) = \mathrm{X}$.

Proof. By Lemma 1, $\overline{f(\mathrm{X})} = \mathrm{X}$ and by (11) and Theorem 3, $f(\mathrm{X})$ is closed in X. Thus $f(\mathrm{X}) = \mathrm{X}$.

REMARK 3. It is easily seen that the assumption of completeness of X in the above Theorem cannot be omitted or even replaced by the assumption that X is totally bounded. In case that X is totally bounded assumption (9) is trivially satisfied and one obtains that a mapping $f: \mathrm{X} \to \mathrm{X}$ of a metric compact space into itself satisfying (11) maps X onto itself. This result was proved by A. Lindenbaum in [8].

We introduce now the following

DEFINITION 4. *Let* P *be a set of pseudometrics on* $\mathrm{X} \times \mathrm{X}$. *A mapping* $f: \mathrm{X} \to \mathrm{X}$ *is a* (λ, P)*-mapping at* $x \in \mathrm{X}$ *iff for each* $p \in \mathrm{P}$ *there exist* $r_p = r_p(x) > 0$ *and* $\lambda_p = \lambda_p(x) > 0$ *such that* $[p(x, y) < r_p] => [p(f(x), f(y)) \leq \leq \lambda_p\, p(x, y)]$.

The function $r_p = r_p(x)$ will be called the r_p–radius and $\lambda_p = \lambda_p(x)$ the λ_p–coefficient. The r_p–radius $r_p = r_p(x)$ is called λ_p–bounded at x iff $\frac{\lambda_p r_p(x)}{r_p(f(x))} \leq 1$. The λ_p–coefficient $\lambda_p = \lambda_p(x)$ is said to form a convergent series at x (is *cs* at x) iff $\sum_{n=1}^{\infty} \prod_{j=1}^{n} \lambda_p(f^j(x)) \leq \infty$, where $f^j(x) = \underbrace{ff \cdots f}_{j}(x)$ denotes the j–*th* iterate of f at the point x ($f^0(x) = x$). Finally an r_p–chain $\mathrm{C}_p(x, x')$ joining the points x and x' is a finite sequence of points $x = x_0, x_1, \cdots, x_n = x'$, $n = n(p, x, x')$ such that for every $i = 0, 1, \cdots, n-1$ one has $p(x_i, x_{i+1}) < r_p(x_i)$.

THEOREM 6. *Let* $f: \mathrm{X} \to \mathrm{X}$ *be a* (λ, P)*–mapping of a complete uniform, Hausdorff space* X *with uniformity generated by the set* P *of pseudometrics having the following properties*:

(a) *there exists a point* x_0 *such that for each* $p \in \mathrm{P}$ *there exists an integer* $l = l(p) \geq 0$ *and a finite sequence of points* $x_0, x_1, \cdots, x_{n(p)} = f(x_0)$ *for which the sequence* $y_0, y_1, \cdots, y_{n(p)}$ *is an* r_p*–chain* $\mathrm{C}_p(y_0, y_n)$, *where* $y_i = f^l(x_i)$, $i = 0, 1, \cdots, n$;

(b) *for every* $p \in \mathrm{P}$ *the* r_p*–radius* $r_p = r_p(x)$ *is* λ_p*–bounded at each point*

$$x = f^j(x_i), \; i = 0, 1, \cdots, n-1, \qquad j = 0, 1, \cdots,$$

and

(c) *the* λ_p*–coefficient is cs at each point* x_i, $i = 0, 1, \cdots, n-1$.

Then the sequence $x_0, f(x_0), ff(x_0) \cdots$ *converges to a fixed point* $\xi = f(\xi)$.

Moreover, if

(a') for every $p \in \mathrm{P}$ each two points x and x' can be joined by an r_p–chain $\mathrm{C}_p(x, x')$;

(b') for every $p \in \mathrm{P}$ the r_p–radius $r_p = r_p(x)$ is λ_p–bounded at each point $x \in \mathrm{X}$,

(c') the λ_p–coefficient is *cs* at each point $x \in \mathrm{X}$, then for every x_0 the sequence: $x_0, f(x_0), ff(x_0), \cdots$ converges to a unique fixed point $\xi = f(\xi)$.

Proof. The proof is a standard one.

COROLLARY 2. *Let* $\mathrm{F}: \mathrm{X} \to \mathbf{X}$ *be a mapping of a complete* linear *Hausdorff space* X *into itself satisfying*:

(12) *there exists* $x_0 \in \mathrm{X}$, $\alpha = \alpha(x_0) \neq 0$, $\delta = \delta(x_0) > 0$ *and a finite number of pseudometrics* $p_{i_1}, \cdots, p_{i_n}$ *(belonging to the set* P *of pseudometrics generating the uniformity of* X) *such that (a)–(c) hold for* $f_y(x) = x - \alpha(x_0)(\mathrm{F}(x) - y)$ *and every* y *with* $\max\limits_{j=1,\cdots,n} p_{i_j}(\mathrm{F}(x_0), y) < \delta$.

Then there exists an open neighborhood U *of* $\mathrm{F}(x_0)$ *such that* $\mathrm{U} \subset \mathrm{F}(\mathrm{X})$.

Proof. Take $y \in \mathrm{U} = \bigcap\limits_{j=1}^{n} \mathrm{B}_{p_{i_j}}(\mathrm{F}(x_0), \delta)$, where $\mathrm{B}_p(y_0, r) = \{y\,; p(y_0, y) < r\}$ (the p–ball with center y_0 and radius r). By Theorem 6 there exists a point $\xi = \xi(y)$ such that $f_y(\xi) = \xi$. Hence by $\alpha(x_0) \neq 0$ it follows that $\mathrm{F}(\xi) = y$.

The proof of the following simple theorem is similar to that of Theorem 2 in [16], p. 1400 and will be omitted.

THEOREM 7. *Let* $\mathrm{F}: \mathrm{X} \to \mathrm{Y}$ *be a polynomial mapping of a complete uniform space* X *into a uniform Hausdorff space* Y *such that* $\mathrm{Int}\,\mathrm{F}(\mathrm{X}) \neq 0$. *Let* $\mathrm{E} \subset \mathrm{X}$ *be the set of points at which* F *is* not open (i.e. $\mathrm{F}(x) \notin \mathrm{Int}\,\mathrm{F}(\mathrm{X})$ *for* $x \in \mathrm{E}$) and *suppose that* $\mathrm{Y} \setminus \mathrm{F}(\mathrm{E})$ *is connected. Then* $\mathrm{F}(\mathrm{X}) = \mathrm{Y}$.

We give three applications and remarks.

APPLICATION 6. Let $\mathrm{F}: \mathrm{X} \to \mathrm{X}$ be a polynomial mapping of a complete linear Hausdorff space X into itself such that property (12) of Corollary 2 holds for every point $x \in \mathrm{X} \setminus \mathrm{E}$ where $\mathrm{X} \setminus \mathrm{F}(\mathrm{E})$ is connected and suppose that (12) holds. Then $\mathrm{F}(\mathrm{X}) = \mathrm{X}$.

Proof. By Corollary 2 the assumptions of Theorem 7 hold (with Y replaced by X). Thus $\mathrm{F}(\mathrm{X}) = \mathrm{X}$.

REMARK 4. To see that Application 6 is in fact a generalization of the fundamental theorem of algebra it suffices to note that a complex polynomial $\mathrm{F}: \mathrm{X} \to \mathrm{X}$ (here X denotes the complex plane) is a polynomial mapping, to use property (12) of Corollary 2 for every $x_0 \in \mathrm{X}$ with $\mathrm{F}'(x_0) \neq 0$, $\alpha(x_0) = \dfrac{1}{\mathrm{F}'(x_0)}$, $\delta = \delta(x_0)$ such that if $x, y \in \mathrm{B}(x_0, \delta)$ then $\left| \dfrac{\mathrm{F}(x) - \mathrm{F}(y)}{x - y} \dfrac{1}{\mathrm{F}'(x_0)} - 1 \right| < \dfrac{1}{2}$ and to note that the set E of points x with $\mathrm{F}'(x) = 0$ is finite and thus $\mathrm{F}(\mathrm{E})$ does not disconnect the plane X (Compare, Remark 3 in [16], p. 1404). This shows the important role played by *the dimension* of the plane for the validity of the fundamental theorem of algebra (A finite set does not disconnect the complex plane (it does disconnect (if $\neq \varnothing$) the real line)).

REMARK 5. Theorem 7 enables to give also the following proof of the fundamental theorem of algebra. Let $\mathrm{F}: \mathrm{X} \to \mathrm{X}$ be a polynomial and $\mathrm{X} = \mathrm{Y}$ be the complex plane. Then F is a polynomial mapping. Now $|\mathrm{F}'(x)|^2$—the Jacobian of F—vanishes in a finite set E of points. Thus $\mathrm{X} \setminus \mathrm{F}(\mathrm{E})$ is connected and by Theorem 7 we have $\mathrm{F}(\mathrm{X}) = \mathrm{X}$.

This proof of the fundamental theorem of algebra was given in [15], p. 160 and [16], p. 1401. The same idea can be, as easily seen, used to prove the existence of solutions for equations with quaternion coefficients (The "fundamental theorem of algebra for quaternions" was proved in [13] and in a more general form in [2]). Let us only illustrate this last statement on the polynomial $f = x^2 + 2bx + c$, where $x = x_1 + x_2 i + x_3 j + x_4 k$, $b = b_1 + b_2 i + b_3 j + b_4 k$ and $c = c_1 + c_2 i + c_3 j + c_4 k$ are quaternions with x_i, b_i and c_i — real. Writing $f = f_1 + f_2 i + f_3 j + f_4 k$ we obtain

$$f_1 = x_1^2 - x_2^2 - x_3^2 - x_4^2 + 2 b_1 x_1 - 2 b_2 x_2 - 2 b_3 x_3 - 2 b_4 x_4 + c_1$$

$$f_2 = 2 x_1 x_2 + 2 b_2 x_1 + 2 b_1 x_2 + 2 b_3 x_4 - 2 b_4 x_3 + c_2 ,$$

$$f_3 = 2 x_1 x_3 + 2 b_3 x_1 + 2 b_4 x_2 + 2 b_1 x_3 - 2 b_2 x_4 + c_3$$

and

$$f_4 = 2 x_1 x_4 + 2 b_4 x_1 - 2 b_3 x_2 + 2 b_2 x_3 + 2 b_1 x_4 + c_4 .$$

For this mapping of the four dimensional Euclidean space E^4 into itself the Jacobian equals to 16 multiplied by

$$(x_1 + b_1)^2 [(x_1 + b_1)^2 + (x_2 + b_2)^2 + (x_3 + b_3)^2 + (x_4 + b_4)^2 + b_2^2 + b_3^2 + b_4^2] + \\ + [(x_2 + b_2) b_2 + (x_3 + b_3) b_3 + (x_4 + b_4) b_4]^2 .$$

Comparing this expression to zero we get

$$x_1 + b_1 = 0 \qquad \text{and} \quad (x_2 + b_2) b_2 + (x_3 + b_3) b_3 + (x_4 + b_4) b_4 = 0 .$$

Thus in case that one of the numbers b_2, b_3, b_4 is $\neq 0$ the set E is a 2-dimensional plane in E^4 and $f(E)$ does not disconnect E^4. In case that

$$b_2 = b_3 = b_4 = 0 \qquad E = \{x ; x_1 = -b_1\}$$

and

$$f(E) = \{f ; f_1 = -b_1^2 - x_2^2 - x_3^2 - x_4^2 + c_1 , f_2 = c_2 , f_3 = c_3 , f_4 = c_4\}$$

evidently does not disconnect E^4. Since a polynomial ($\neq$ constant) with quaternion coefficients maps sequences tending to ∞ onto sequences tending to ∞, it is a polynomial mapping. Thus by theorem 7 one has $f(E^4) = E^4$.

REMARK 6. A proof of the fundamental theorem of algebra using similar ideas can be found in [9] (See also [14], p. 390). The proofs mentioned in Remarks 4 and 5 seem to be "less topological" (for example the notion of a compactification is not used there). In fact the only properties used in these proofs are:

(i) the continuity of a complex polynomial,

(ii) the connectedness of the Euclidean plane and of the Euclidean plane without a finite set of points and the local compactness,

(iii) the fact that a non constant complex polynomial maps sequences tending to ∞ onto sequences tending to ∞,

(iv) the inverse function theorem,

(this last property can be proved (See Remark 4) by using the contraction mapping principle) and

(v) the fact that a complex polynomial has at most a finite number of roots.

APPLICATION 7. Let $F : X \to Y$ be a polynomial mapping of a separable Banach space X into an infinite dimensional Banach space Y such that $\operatorname{Int} F(X) \neq \varnothing$ and suppose that

$$[x \in X \text{ and } F(x) \notin \operatorname{Int} F(X)] \tag{13}$$

implies the existence of an open neighborhood $U = U_x$ of x such that $F(U)$ is contained in a finite dimensional plane $Q_x \subset Y$ (the dimension of Q_x may depend on x).

Then $F(X) = Y$.

Proof. Let J_1 be the set of all points $x \in X$ such that for some neighborhood $U = U_x$, $F(U)$ is contained in a finite dimensional plane $Q_x \subset Y$. By (13) and Theorem 3 of [3] the set $J = F(J_1)$ does not disconnect Y (i.e., $Y \setminus J$ is connected). It remains to apply Theorem 7.

In the following application which can be proved also directly the notion of polynomiality of the mapping is not used.

APPLICATION 8. Let $\varphi(t, u)$ be a real valued continuous function $-\infty < t, u < \infty$ such that there exist functions $\alpha(t)$ and $\beta(t)$ with $0 < \alpha(t) < \varphi_u(t, u) < \beta(t)$ for all $-\infty < u < \infty$ (no continuity or boundedness assumptions are made about $\alpha(t)$, $\varphi_u(t, u)$, $\beta(t)$). Then there exists a unique continuous function $\xi(t)$, $-\infty < t < \infty$ such that $\varphi(t, \xi(t)) = 0$ or all $-\infty < t < \infty$.

Proof. Consider the linear (real) space X of all continuous real valued functions $x = x(t)$, $-\infty < t < \infty$ with the uniformity generated by the set p of pseudometrics (even seminorms) p_t defined by $p_t(x, y) = |x(t) - y(t)|$ for $x, y \in X$, where $-\infty < t < \infty$. The space X is clearly complete uniform Hausdorff (not metrizable).

Define $f: X \to X$ by $f(x(t)) = x(t) - \dfrac{1}{1 + \beta(t)} \varphi(t, x(t))$. For a fixed t denote $m = \alpha(t)$ and $M = \beta(t)$. Then

$$p_t(f(x), f(y)) = \left| x(t) - y(t) - \frac{\varphi_u(t, \bar{u})}{1 + M} (x(t) - y(t)) \right| \leq \left(1 - \frac{m}{1 + M}\right) p_t(x, y).$$

Putting $\lambda_{p_t}(x) = 1 - \dfrac{m}{1 + M}$ it is trivially seen that $f: X \to X$ is a (λ, P) mapping satisfying the assumptions (a') — (c') of Theorem 6 ($r_p = r_{p_t} = \infty$ for every $p_t \in P$). Thus by Theorem 6 there exists a unique fixed point $\xi = f(\xi) \in X$. It follows that the unique function $\xi(t) \in X$ satisfies $\varphi(t, \xi(t)) = 0$ for all $-\infty < t < \infty$.

We conclude the paper with two problems.

PROBLEM 1. It can be easily shown that if $A = A(x_0): X \to X$ is a *linear* mapping of a Banach space X into itself such that for points x, y of the ball $B(x_0, r)$ one has $\|x - y - A(f(x) - f(y))\| < \alpha \|x - y\|$ with $\alpha < 1$, $x \neq y$, where $f: X \to X$ is a mapping of X into itself, then $f(B(x_0, r))$ contains a ball $B(f(x_0), r_0)$ with radius $r_0 = \frac{1 - \alpha}{\|A\|} r$. In Remark 4 this property together with the fact that the set of roots of a polynomial does not disconnect the plane was used for $Ax = \frac{1}{F'(x_0)} x$ to prove the fundamental theorem of algebra. It can be easily shown that the set E of roots of a polynomial f with quaternion coefficients can be covered by a *finite number of 2-dimensional spheres* and thus $f(E)$ does not disconnect the four dimensional Euclidean space E^4. Using similar ideas as in Remark 4 and the notion of a differential of a quaternion function (See [4], p. 430) prove the "fundamental theorem of algebra for quaternions" (See [13] and [2]).

PROBLEM 2. Let $f: E \to Y$ be a mapping of a subset E of a Banach space X into a Banach space Y. Find non-trivial conditions on E and f under which $Y \setminus f(E)$ is connected (It follows easily from the results in [3] that if f is completely continuous and Y is infinite dimensional then $Y \setminus f(E)$ is connected for every E).

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