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JÓZEF BLASS, WŁODZIMIERZ HOLSZTYŃSKI

Cubical polyhedra and homotopy

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Topologia. — *Cubical polyhedra and homotopy.* Nota di JÓZEF BLASS e WŁODZIMIERZ HOLSZTYŃSKI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Per ogni spazio topologico, X, viene assegnata una costruzione funtoriale di un complesso cubico QX. Quando X è compatto, QX risulta equivalente ad X (a meno di un'omotopia), ed è una dualizzazione del semisimpliciale S(X). Di tutto ciò verranno fatte numerose applicazioni in lavori successivi.

In the present paper we define the category QP of cubical polyhedra and we introduce for a topological space X the cubical polyhedron QX. In the case of a compact space X, the homotopy types of X and QX are shown to be naturally equivalent. More precisely:

Let Ht be the homotopy functor, let Q be the functor assigning QX to X and let $F_0 : QP \to Top$ be the "forgetful" functor. We construct a natural transformation $\Gamma : Ht \circ F_0 \circ Q \to Ht$, and we show that Γ restricted to compact spaces is a natural equivalence. This is a dualization of semi-simplicial S(X) and of the natural weak homotopy equivalence $S(X) \to X$. We also show, that QX can be represented as the limit of an inverse system of finite polyhedra. This was used to construct a homology theory of the Čech type built on the cubical scheme. We will present this construction in forthcoming papers.

Throughout this paper we will use the following *notations*:

Id_A – the identity map of A onto A;

$f \circ g$ – set-theoretical composition of f and g (composition in the category of sets);

$f \bullet g$ – composition of f and g in a category;

$I = [-1 ; 1]$ – the set of real x such that $-1 \leq x \leq 1$;

$F_{(a,\varepsilon)} = \{x = (x_\alpha)_{\alpha \in A} I^A : x_a = \varepsilon\}$;

Top – the topological category of pairs;

Ht is the homotopy functor and HtTop is the category of topological pairs with homotopy classes of mappings as morphisms. We will often use $[f]$ to denote $Ht(f)$.

1. THE CUBICAL CATEGORY QP

Let $\beta : B \to \{-1 , 1\}$ be a function defined on a subset B of A. Then define

$$F_\beta = \bigcap_{a \in B} F_{(a,\beta(a))}$$

and we call F_β a face of I^A. If $B = A$ then F_β is a single element set consisting of a vertex of I^A.

(*) Nella seduta del 20 febbraio 1971.

If $B = \varnothing$ then we define $F_\beta = I^A$.

All faces are non-empty sets. If F_β, F_γ are faces of I^A, $\beta : B \to \{-1, 1\}$, $\gamma : C \to \{-1, 1\}$, $B \cup C \subset A$, then

$$F_\beta \subset F_\gamma \quad \text{iff} \quad C \subset B \quad \text{and} \quad \gamma = \beta \mid C .$$

If $X \subset I^A$ then carr X will denote the smallest face containing X. In the case of a single element set $X = \{x\}$ we will write carr x instead of carr $\{x\}$.

(1.1). DEFINITION. A subset W of I^A is said to be a cubical polyhedron (or simply a polyhedron) iff

$$W = \bigcup \{\text{carr } x : x \in W\} .$$

A pair (W, V) consisting of polyhedra $V \subset W \subset I^A$ is said to be a polyhedral pair.

Thus every polyhedron $W \subset I^A$ is a union of faces of I^A. Obviously, the converse is also true. Every union of a family of faces of I^A is a polyhedron.

We will always identify a pair $(W, \varnothing)$ and W.

(1.2) THEOREM. *Let* $W \subset I^A$ *be a polyhedron and let* F_β *be a face of* I^A, *where* $\beta : B \to \{-1, 1\}$ *for a subset* B *of* A. *Then*

$$F_\beta \subset W \quad \textit{iff} \quad \bigcap \{G_{(a,\varepsilon)} : a \in A \setminus B \quad \textit{or} \quad \beta(a) = \varepsilon\} \neq \varnothing$$

where $G_{(a,\varepsilon)} = W \setminus F_{(a,-\varepsilon)}$ *for* $a \in A$, $\varepsilon = \pm 1$.

Proof. Suppose that $F_\beta \subset W$. Let

$$x_a = \begin{cases} 0 & \text{for} \quad x \in A \setminus B \\ \beta(a) & \text{for} \quad x \in B . \end{cases}$$

Obviously $x \in G_{(a,\varepsilon)}$ if $a \in A \setminus B$ or $\beta(a) = \varepsilon$. Thus

(i) $$x \in \bigcap \{G_{(a,\varepsilon)} : a \in A \setminus B \quad \text{or} \quad \beta(a) = \varepsilon\} \neq \varnothing .$$

Conversely, if (i) holds for an $x = (x_a)_{a \in A} \in I^A$, then $|x_a| \neq 1$ for $a \in A \setminus B$ and $x_a \neq -\beta(a)$ for $a \in B$.

Let $F_\gamma = \text{carr } x$ for a function $\gamma : C \to \{-1, 1\}$. Then $C \subset B$ and $\gamma = \beta \mid C$. Thus $F_\beta \subset F_\gamma \subset W$.

(1.3) DEFINITION. Given polyhedral pairs (W,V) and (W_1, V_1), $V \subset W \subset I^A$ and $V_1 \subset W_1 \subset I^{A_1}$. A cubical morphism $q : (W,V) \to (W_1, V_1)$ is a function $q : A_1 \to A$ such that $f_q(W) \subset W_1$ and $f_q(V) \subset V_1$, where $f_q : I^A \to I^{A_1}$ is a map given by $(f_q(x))_a = x_{q(a)}$ for every $a \in A_1$ and $x = (x_\alpha)_{\alpha \in A} \in I^A$. The composition $p \circ q : (W, V) \to (W_2, V_2)$ of cubical morphisms $q : (W,V) \to (W_1, V_1)$ and $p : (W_1, V_1) \to (W_2, V_2)$ is defined by

$$p \circ q = q \bullet p .$$

The unit morphism $1_{(W,V)}$ is defined by

$$1_{(W,V)} = \mathrm{Id}_A \qquad (\text{for} \quad V \subset W \subset I^A)\,.$$

Thus $f_{1_{(W,V)}} = \mathrm{Id}_{I^A}$.

(1.4) DEFINITION. The cubical category QP is the category of polyhedral pairs and cubical morphisms of such pairs.

In the next sections we will deal with "forgetful" functor $F_0 : QP \to \mathrm{Top}$ and the functor $F = \mathrm{Ht} \circ F_0 : QP \to \mathrm{HtTop}$. These functors are given by

$$F_0(W, V) = F(W, V) = (W, V) \tag{1.5}$$

for every polyhedral pair (W, V), and

$$F_0(q) = f_q \qquad (\text{see def. } (1.3)) \tag{1.6}$$

and

$$F(q) = \mathrm{Ht}(f_q) = [f_q]\,. \tag{1.7}$$

The following simple propositions will be useful.

(1.8) PROPOSITION. *Let* $x = (x_a)_{a \in A} \in I^A$ *and* $B \subset A$ *and* $\beta : B \to \{-1, 1\}$. *Then carr* $x = F_\beta$ *iff* $B = \{a \in A : |x_a| = 1\}$ *and* $\beta(a) = x_a$ *for every* $a \in B$.

(1.9) PROPOSITION. *Let* $g : (W, W_0) \to (V, V_0)$ *be a* 1 — 1 *continuous map of a polyhedral pair* (W, W_0) *onto* (V, V_0) (*hence* $g(W) = V$ *and* $g(W_0) = V_0$) *and let* g *be induced by a cubical morphism* $q : (W, W_0) \to (V, V_0)$, *i.e. let* $g = f_q = F_0(q)$. *Then* q *is a cubical isomorphism of* (W, W_0) *and* (V, V_0) *and the inverse function* g^{-1} *is induced by* q^{-1}, $g^{-1} = F_0(q^{-1})$ (*Hence* g *is a homeomorphism*).

2. THE CUBICAL NERVES

Let X be a topological space and let $T = T(X)$ be the family of all functionally open subsets of X [1]. We will also use the following notation

$$A(X) = \{G = (G_{-1}, G_1) \in T \times T : G_{-1} \cup G_1 = X\}\,,$$

and we define $\pi_\varepsilon : A(X) \to T$ as

$$\pi_\varepsilon(G) = G_\varepsilon \quad \text{for every} \quad G = (G_{-1}, G_1) \in A(X) \quad , \quad \varepsilon = \pm 1\,.$$

Next, for $B \subset A \subset A(X)$, $\beta : B \to \{-1, 1\}$ and $X_0 \subset X$ we put

$$\operatorname{supp} F_\beta = \bigcap \{\pi_\varepsilon(G) : G \in A \setminus B \quad \text{or} \quad \beta(G) = \varepsilon\} \tag{2.1}$$

(1) I.e. of sets of the form $f^{-1}(R \setminus \{0\})$, where $f : X \to R$ is a continuous real-valued function.

and

$$\text{supp}_{X_0} F_\beta = X_0 \cap \text{supp } F_\beta . \tag{2.2}$$

We define also the nerve $N_{X_0}A$ of $A \subset A(X)$ in $X_0 \subset X$, as a subpolyhedron of I^A such that

$$F_\beta \subset N_{X_0}A \quad \text{iff} \quad \text{supp}_{X_0} F_\beta \neq \varnothing . \tag{2.3}$$

For X and (X, X_0) we put

$$NA = N_X A \quad \text{and} \quad NA(X, X_0) = (NA, N_{X_0}A) . \tag{2.4}$$

(2.5) PROPOSITION. *Let* $X_0 \subset X$ *and* $B \subset A \subset A(X)$. *Then the inclusion map* $i_B^A : B \to A$ *is a cubical morphism from* $NA(X, X_0)$ *into* $NB(X, X_0)$.

Proof. The map $f_i : I^A \to I^B$ for $i = i_B^A$ (see def. (1.3)) is the projection $p_B^A : I^A \to I^B$. Let $x = (x_G)_{G \in A} \in N_{X_0}A$. Then carr $x \subset N_{X_0}A$. By Prop. (1.8) that means

$$X_0 \cap \bigcap \{\pi_\varepsilon(G) : x_G \neq -\varepsilon \text{ and } G \in A\} \neq \varnothing .$$

Thus

$$X_0 \cap \bigcap \{\pi_\varepsilon(G) : x_G \neq -\varepsilon \text{ and } G \in B\} \neq \varnothing ,$$

and $p_B^A(x) = (x_a)_{a \in B} \in N_{X_0}B$. Thus $p_B^A(N_{X_0}A) \subset N_{X_0}B$ for any $X_0 \subset X$. In particular, $p_B^A(NA) \subset NB$. Thus i_B^A is a cubical morphism of $NA(X, X_0)$ into $NB(X, X_0)$.

Obviously

$$\text{if} \quad C \subset B \subset A \subset A(X) \quad \text{then} \quad i_C^A = i_C^B \circ i_B^A . \tag{2.6}$$

(2.7) PROPOSITION. *Let* $g : (X, X_0) \to (Y, Y_0)$ *be a continuous map of topological pairs and let* $A \subset A(X)$, $B \subset A(Y)$ *be such sets that* $g^{-1}(B) \subset A$. *Then the function* $N_B^A(g) : A_1 \to A$ *given by* $(N_B^A(g))(G) = g^{-1}(G)$ (2) *is a cubical morphism of* $NA(X, X_0)$ *into* $NB(Y, Y_0)$.

The proof is similar to the proof of Prop. (2.5) (which is a special case of the above (2.7)).

Obviously

If composition of continuous maps $g \bullet f$ is defined as well as $N_B^A(f)$ and $N_C^B(g)$ then $N_C^A(g \bullet f)$ is defined and

$$N_C^A(g \bullet f) = N_C^B(g) \circ N_B^A(f) . \tag{2.8}$$

(2) $g^{-1}(B) = \{g^{-1}(G) : G \in B\}$, where $g^{-1}(G) = (g^{-1}(G_{-1}), g^{-1}(G_1))$ for $G = (G_{-1}, G_1)$.

3. THE CUBICAL FUNCTOR Q

Let FinK denote the family of all finite subsets of K.

(3.1) DEFINITION. Let X_0 be a subspace of X. We define

$$Q_X X_0 = \bigcap\{(p_A^{A(X)})^{-1}(N_{X_0}A) : A \in \operatorname{Fin} A(X)\},$$
$$QX = Q_X Q \quad \text{and}$$
$$Q(X, X_0) = (QX, Q_X X_0).$$

Evidently

(3.2) $$x = (x_G)_{G \in A(X)} \in Q_X X_0 \quad \text{iff}$$

$X_0 \cap \{\pi_\varepsilon(G) : x_G \neq -\varepsilon \text{ and } G \in A\} \neq \emptyset$ for every $A \in \operatorname{Fin} A(X)$.

(3.3) PROPOSITION. *Let $f:(X, X_0) \to (Y, Y_0)$ be a continuous map. Then the function $Q(f): A(Y) \to A(X)$ given by $(Q(f))(G) = f^{-1}(G)$ for every $G \in A(Y)$ is a cubical morphism of $Q(X, X_0)$ into $Q(Y, Y_0)$.*

(3.4) PROPOSITION. *If composition $g \bullet f$ of continuous maps of topological pairs is defined then $Q(g) \circ Q(f)$ is defined and $Q(g) \circ Q(f) = Q(g \bullet f) = Q(f) \bullet Q(g)$.*

We have also

(3.5) $$Q(\mathrm{Id}_{(X,X_0)}) = 1_{Q(X,X_0)}.$$

Thus we have obtained a functor $Q : \mathrm{Top} \to \mathrm{QP}$. We call this functor the cubical functor. Functor Q has the following important properties, which are direct consequences of def. (3.1) and Prop. (3.3).

(3.6) PROPERTY. $(Q(X, X_0), p_B^{A(X)} : B \in \operatorname{Fin} A(X))$ is a representation of $Q(X, X_0)$ as the inverse limit of the system

$$(N_B^A(X, X_0), \; p_B^A : B \subset A \in \operatorname{Fin} A(X))$$

in the category Top ($p_B^A = F_0(i_B^A)$ denotes the projection map $I^A \to I^B$ induced by the inclusion $i_B^A : B \to A$, $B \subset A$).

(3.7) PROPERTY. Let $g:(X, X_0) \to (Y, Y_0)$ be a continuous map. Then the map $f_{Q(g)} = F_0 \circ Q(g) : Q(X, X_0) \to Q(Y, Y_0)$ is the inverse limit map in Top of maps $g_B^A : NA(X, X_0) \to NB(Y, Y_0)$, $A \supset f^{-1}(B)$, $g_B^A = F_0(N_B^A g)$.

Using Prop. (1.9), we obtain from the above two properties the following analogous properties in Top

(3.8) PROPERTY. $(Q(X, X_0), i_B^{A(X)} : B \in \operatorname{Fin} A(X))$ is a representation of $Q(X, X_0)$ as the inverse limit of the system $(N_B^A(X, X_0), i_B^A : B \subset A \in \operatorname{Fin} A(X))$ in the cubical category QP.

(3.9) PROPERTY. Let $g : (X, X_0) \to (Y, Y_0)$ be a continuous map. Then $Q(g) : Q(X, X_0) \to Q(Y, Y_0)$ is the inverse limit morphism of morphisms $N_B^A g (f^{-1}(B) \subset A \in \mathrm{Fin}\, A(X)$ and $B \in \mathrm{Fin}\, A(Y))$ in the cubical category QP.

4. THE NATURAL TRANSFORMATION $\Phi : Ht \to F \circ Q$

To each pair $G = (G_{-1}, G_1) \in A(X)$ we assign a continuous function $f_G : X \to I$ such that

(4.1) $$f_G \mid X \setminus G_\varepsilon = -\varepsilon \quad \text{for} \quad \varepsilon = \pm 1 .$$

We also define the continuous function $f = f^X : X \to QX \subset I^{A(X)}$ as $f(x) = (f_G(x))_{G \in A}$ i.e. $f = \underset{G \in A(X)}{\Delta} f_G$.

The following proposition shows that the mapping f is well defined.

(4.2) PROPOSITION. $f(X, X_0) \subset Q(X, X_0)$ *i.e.* $f(X) \subset QX$ *and* $f(X_0) \subset Q_X X_0$.

Proof. We will show that $f(X_0) \subset Q_X X_0$. Set $A \in \mathrm{Fin}\, A(X)$ and suppose that $x \in X_0$. Note that if $f_G(x) \neq -\varepsilon$ then $x \in G_\varepsilon$.

Therefore

$$X_0 \cap \{\pi_\varepsilon(G) : (f(x))_G \neq -\varepsilon \quad \text{and} \quad G \in A\} \neq \emptyset .$$

Hence (see (3.2)) $f(x) \in Q_X X_0$. Thus $f(X_0) \subset Q_X(X_0)$. In particular $f(x) \subset Q_X(X) = QX$.

The following assertion is evident.

(4.3) PROPOSITION. *Let* $f' = \underset{G \in A}{\Delta} f'_G : X \to X \subset I^A$ *by another mapping such that* $f'_G \mid X \setminus G_\varepsilon = -\varepsilon$ *for every* $G \in A$ *and* $\varepsilon = \pm 1$. *Then* $f \simeq f' : (X, X_0) \to Q(X, X_0)$ *and we have a canonical homotopy* $h : (X, X_0)\ I \to Q(X, X_0)$ *given by the formula*

$$h(x, t) = (1 - t) f(x) + t f'(x) \quad \text{for} \quad t \in I \quad \text{and} \quad x \in X .$$

Proof. Indeed, for $f_t = \underset{G \in A}{\Delta} f_{t,G}$, given by $f_t(x) = h(x, t)$, $f_{t,G} \mid X \setminus G_\varepsilon = \varepsilon$ (see (4.1)).

Thus the homotopy class of $f : (X, X_0) \to Q(X, X_0)$ which satisfies (4.1), depends only on (X, X_0), and we can define $\Phi : Ht \to F \circ Q$ by putting

$$\Phi(X, X_0) = [f] .$$

(4.4) THEOREM. $\Phi : Ht \to F \circ Q$ *is a natural transformation.*

Proof. Let $\varphi : (X, X_0) \to (Y, Y_0)$ be a continuous mapping and let $\Phi(Y, Y_0) = [g]$ for some $g = \underset{G \in A(Y)}{\Delta} g_G$ such that (4.1) holds for every g_G and $G \in A(Y)$. Then let

$$f_{\varphi^{-1}(G)} = g_G \circ \varphi \quad \text{for every} \quad G \in A(Y)$$

and let f_G for $G \in A(X) \setminus \varphi^{-1}(A(Y))$, be a mapping such that (4.1) holds. Then for $f = \underset{G \in A(X)}{\Delta} f_G$ we have $\Phi(X, X_0) = [f]$ and $g \circ \varphi = (F_0 \circ Q)(\varphi) \circ f$. Thus $[g] \circ [\varphi] = (F \circ Q)(\varphi) \circ [f]$ and finally

$$\Phi(Y, Y_0) \circ Ht(\varphi) = (F \circ Q)(\varphi) \circ \Phi(X, X_0).$$

5. The natural transformation $\Gamma_0 : F_0 \circ Q \mid \mathrm{Comp} \to 1_{\mathrm{Comp}}$

For a compact space X we define the function g as $g = \Gamma_0(X) : QX \to X$.

Let $x = (x_G)_{G \in A} \in QX$ and let $\mathcal{F}(x)$ be the filter in the lattice $T = T(X)$ generated by the family

$$(5.1) \qquad B(x) = \{\pi_\varepsilon(G) : x_G \neq -\varepsilon \,, \quad G \in A(X)\}.$$

Thus, for every $G \in A(X)$, we have either $G_{-1} \in \mathcal{F}(x)$ or $G_1 \in \mathcal{F}(x)$. In other words, $G_{-1} \in \mathcal{F}(x)$ or $G_1 \in \mathcal{F}(x)$ for every G_{-1}, $G_1 \in T$ such that $G_{-1} \cup G_1 = X$. Thus, $\mathcal{F}(x)$ has the unique limit in X. We define $g : QX \to X$ by the formula

$$(5.2) \qquad g(x) = \lim \mathcal{F}(x).$$

(5.3) LEMMA. *$g : QX \to X$ is a continuous mapping.*

Let $x = (x_G)_{G \in A} \in QX$ and let V be a neighborhood of $g(x)$ in X. Let $\varphi : X \to I$ be a continuous mapping such that $\varphi(g(x)) = 1$ and $\varphi \mid X \setminus V = -1$. Then for $G_{-1} = \varphi^{-1}([-1\,;\,1/2))$ and $G_1 = \varphi^{-1}((-1/2\,;\,1])$ we have that $G = (G_{-1}, G_1) \in A$ and $g(x) \notin \bar{G}_{-1}$ and $\bar{G}_1 \subset V$. Thus $G_{-1} \notin \mathcal{F}(x)$ and $G_1 \in \mathcal{F}(x)$. The set $U = \{y = (y_H)_{H \in A} \in QX : y_G > -1\}$ is a neighborhood of x in QX ($x_G = 1$) such that $g(U) \subset \bar{G}_1 \subset V$. Q.E.D.

(5.4) LEMMA. *Let X_0 be a compact subspace of X. Then $g(Q_X(X_0)) \subset X_0$.*

Proof. Suppose $x = (x_G)_{G \in A(X)} \in Q_X(X_0)$. Then the family $\{\pi_\varepsilon(G) \cap X_0 : G \in A(X)$ and $x_G \neq \varepsilon\}$ has the finite intersection property. Therefore, $g(x) = \lim \mathcal{F}(x) \in X_0$.

Using (4.4) we can consider g as a mapping $g : Q(X, X_0) \to (X, X_0)$ for every compact pair (X, X_0). Let Comp be the category of compact pairs. We have thus obtained a transformation $\Gamma_0 : F_0 \circ Q \mid \mathrm{Comp} \to 1_{\mathrm{Comp}}$ such that $\Gamma_0(X, X_0) = g : Q(X, X_0) \to (X, X_0)$.

(5.5) THEOREM. *$\Gamma_0 : F_0 \circ Q \mid \mathrm{Comp} \to 1_{\mathrm{Comp}}$ is a natural transformation.*

6. Φ and Γ as natural equivalences

In this section we will prove the basic theorem of this paper.

(6.1) THEOREM. *Let $\Gamma : F \circ Q \mid \mathrm{Comp} \to Ht \mid \mathrm{Comp}$ be the natural transformation induced by Γ_0 (see § 5) and let $\Phi \mid \mathrm{Comp} : Ht \mid \mathrm{Comp} \to F \circ Q \mid \mathrm{Comp}$ be the restriction of Φ (see § 4). Then $\Phi \mid \mathrm{Comp} = \Gamma^{-1}$, and consequently Γ and $\Phi \mid C$ are natural equivalences.*

Proof. If $(X, X_0) \in C$ and $[f] = \Phi(X, X_0)$ then

$$\Gamma_0(X, X_0) \circ f = \mathrm{Id}_{(X,X_0)}, \tag{6.2}$$

and for $h(x,t) = (1-t)\,x + t\cdot(f\circ\Gamma_0(X, X_0))(x)$, $x \in Q(X, X_0)$, $t \in I$, if

$$h((X, X_0)\times I) \subseteq Q(X, X_0) \tag{6.3}$$

then

$$f\circ\Gamma_0(X, X_0) \underset{h}{\simeq} 1_{Q(X,X_0)}. \tag{6.4}$$

Thus it suffices to prove (6.3).

Let $x \in Q_X X_0$ and $0 < t < 1$ and let $g = \Gamma_0(X, X_0)$. Then (by (5.1))

$$B(h(x, t)) = B((1-t)\,x + t\cdot(f\circ g)(x)) = B(x) \cup B(f\circ g(x)).$$

Since the family $\{U \cap X_0 : U \in B(x)\}$ has the finite intersection property, and $g(x) \in \overline{U \cap X_0}$ for every $U \in \mathfrak{F}(x)$, and $g(x) \in V$ for every $V \in B(f\circ g(x))$ it follows that $\{U \cap X_0 : U \in B(h(x,t))\}$ has the finite intersection property. Thus, $h(x, t) \in Q_X X_0$. In particular, for $X = X_0$ we have $h(x, t) \in QX$.