Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

U. P. Singh, B. N. Prasad

Special curvature collimations in Finsler space

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **50** (1971), n.2, p. 122–127.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1971_8_50_2_122_0>

Geometria differenziale. — *Special curvature collineations in Finsler space.* Nota di U. P. SINGH e B. N. PRASAD, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Si dice che uno spazio di Finsler possiede una collineazione speciale di curvatura se esiste un campo di vettori rispetto al quale la derivata di Lie del tensore di curvatura secondo Berward sia nulla. Si studiano in questa Nota relazioni fra questa curvatura ed altre simmetrie che possiede lo spazio.

INTRODUCTION

In a recent paper Katzin *et al.* [1] formulated that a Riemannian space V_n admits a symmetry called a curvature collineation provided there exists a vector field v^i such that $\mathfrak{L}_v R^m_{ijk} = 0$ where R^m_{ijk} is the Riemannian curvature tensor (Eisenhart [3]) and $\mathfrak{L}_v$ denotes the Lie derivative (Yano [4]). In [2] we have generalised this idea to a Finsler space and have defined curvature collineation in the manner stated above for Cartan's curvature tensor K^i_{jhk} (Rund [5] page 97). In the present paper we are concerned with a symmetry property (of a Finsler space) which we shall call special curvature collineation. Several theorems establishing relations between this special curvature collineation and the other symmetries admitted by the Finsler space have been obtained.

1. FUNDAMENTAL FORMULAE

Let F_n be an n-dimensional Finsler space equipped with the positively homogeneous metric function $F(x, \dot{x})$ of degree one in $\dot{x}^i$. The metric tensor of F_n is given by

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \quad (1) \tag{1.1}$$

The covariant derivatives, in the sense of Cartan and Berwald, of a vector field $X^i(x, \dot{x})$ are respectively given by (Rund [5] page 74, 80),

$$X^i_{|k},(x, \dot{x}) = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j \Gamma^{*i}_{jk}, \tag{1.2}$$

$$X^i_{(k)}(x, \dot{x}) = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j G^i_{jk}, \tag{1.3}$$

where $\Gamma^{*i}_{jk}(x, \dot{x})$ and $G^i_{jk}(x, \dot{x})$ are respectively the connection co-efficients of Cartan and Berwald. These coefficients are symmetric in their lower indices and are homogenous of degree zero in $\dot{x}^i$.

(*) Nella seduta del 20 febbraio 1971.

(1) $\partial_j X^i = \dfrac{\partial X^i}{\partial x^j}$ and $\dot{\partial}_j X^i = \dfrac{\partial X^i}{\partial \dot{x}^j}$.

The curvature tensors arising from (1.2) and (1.3) are such that (Rund [5] page 97, 125).

(1.4) $$K^i_{jkh}(x, \dot{x}) = (\partial_h \Gamma^{*i}_{jk} - \dot{\partial}_m \Gamma^{*i}_{jk} G^m_h) - (\partial_k \Gamma^{*i}_{jh} - \dot{\partial}_m \Gamma^{*i}_{jh} G^m_k) + \Gamma^{*m}_{jk} \Gamma^{*i}_{hm} - \Gamma^{*m}_{jh} \Gamma^{*i}_{km},$$

(1.5) $$H^i_{jkh}(x, \dot{x}) = (\partial_h G^i_{jk} - G^i_{mjk} G^m_h) - (\partial_k G^i_{jh} - G^i_{mjh} G^m_k) + G^m_{jk} G^i_{hm} - G^m_{jh} G^i_{km}$$

respectively, where

(1.6) $$G^i_{mjk}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_m G^i_{jk} = \dot{\partial}_m \dot{\partial}_j \dot{\partial}_k G^i .$$

The relation between these two curvature tensors is given by (Rund [5] page 126)

(1.7) $$H^i_{jkh} = K^i_{jkh} + \dot{x}^\gamma \dot{\partial}_j K^i_{\gamma kh} .$$

The Weyl projective curvature tensors W^i_h and W^i_{jkh} are (Rund [5] page 140–141)

(1.8 a) $$W^i_h = H^i_h - H\delta^i_h - \frac{1}{n+1} (\dot{\partial}_\gamma H^\gamma_h - \dot{\partial}_h H) \dot{x}^i ,$$

(1.8 b) $$\begin{aligned} W^i_{jkh} &= \frac{1}{3} (\dot{\partial}_j \dot{\partial}_k W^i_h - \dot{\partial}_j \dot{\partial}_h W^i_k) = \\ &= H^i_{jkh} + \frac{1}{n+1} (H_{kh} - H_{hk}) \delta^i_j + \frac{1}{n+1} (\dot{\partial}_j H_{kh} - \dot{\partial}_j H_{hk}) \dot{x}^i + \\ &+ \frac{1}{n^2-1} (nH_{jh} + H_{hj} + \dot{x}^\gamma \dot{\partial}_j H_{h\gamma}) \delta^i_k - \\ &- \frac{1}{n^2-1} (nH_{jk} + H_{kj} + \dot{x}^\gamma \dot{\partial}_j H_{k\gamma}) \delta^i_h \end{aligned}$$

where

(1.9) $$H^i_h = H^i_{jkh} \dot{x}^j \dot{x}^k \quad , \quad H = \frac{1}{n+1} H^i_i \quad \text{and} \quad H_{jk} = H^i_{jki} .$$

We have the following commutation formulae involving a tensor T^i_{jk} and the connection G^i_{jk}

(1.10) $$\mathfrak{L}_v (\dot{\partial}_k T^i_{jh}) - \dot{\partial}_k (\mathfrak{L}_v T^i_{jh}) = 0 ,$$

(1.11) $$(\mathfrak{L}_v G^i_{jh})_{(k)} - (\mathfrak{L}_v G^i_{kh})_{(j)} = \mathfrak{L}_v H^i_{hjk} + (\mathfrak{L}_v G^\gamma_{kl}) \dot{x}^l G^i_{\gamma jh} - (\mathfrak{L}_v G^\gamma_{jl}) \dot{x}^l G^i_{\gamma kh} ,$$

(1.12) $$(\dot{\partial}_k T^i_{jh})_{(l)} = \dot{\partial}_k (T^i_{jh(l)}) + T^i_{j\gamma} G^\gamma_{hkl} + T^i_{\gamma h} G^\gamma_{jkl} - T^\gamma_{jh} G^i_{\gamma kl} .$$

The following definitions and lemmas will be used in this paper. *Motion* (Rund [5] page 175). A F_n is said to admit a motion provided there exists a vector field v^i such that

(1.13) $$\mathfrak{L}_v g_{ij} = 0 .$$

Affine motion (Yano [4] page 190). A F_n is said to admit an affine motion provided there exists a vector field v^i such that

(1.14) $$\mathfrak{L}_v G^i_{jk} = 0 .$$

Projective motion (Sinha [6]). A F_n is said to admit a projective motion provided there exists a vector field v^i such that

$$\mathfrak{L}_v G^i_{jk} = -\delta^i_j P_k - \delta^i_k P_j - P_{jk} \dot{x}^i , \tag{1.15}$$

where

$$P_k = \dot{\partial}_k P \quad , \quad P_{jk} = \dot{\partial}_j \dot{\partial}_k P , \tag{1.16}$$

and $P(x , \dot{x})$ is an arbitrary scalar function positively homogeneous of the first degree in $\dot{x}^i$.

Homothetic motion (Hiramatu [7]). A F_n is said to admit a homothetic motion if there exists a vector field v^i such that

$$\mathfrak{L}_v g_{ij} = 2 \sigma g_{ij} \tag{1.17}$$

holds with σ a non-zero constant.

Curvature collineation (Prasad [2]). A F_n is said to admit a curvature collineation if there exists a vector field v^i such that

$$\mathfrak{L}_v K^i_{jkh} = 0 . \tag{1.18}$$

Ricci collineation (Prasad [2]). A F_n is said to admit a Ricci collineation if there exists a vector field v^i such that

$$\mathfrak{L}_v K_{jk} = 0 \qquad \text{where} \quad K_{jk} = K^i_{jki} . \tag{1.19}$$

LEMMA (1.1) (Prasad [2]). *In a* F_n *every motion is an affine motion.*

LEMMA (1.2) (Prasad [2]). *In a* F_n *every homothetic motion is an affine motion.*

LEMMA (1.3) (Rund [5] page 147). *The Weyl tensor* W^i_h *vanishes identically in an isotropic Finsler space.*

2. SPECIAL CURVATURE COLLINEATION

The infinitesimal transformation

$$\bar{x}^i = x^i + v^i(x)\, \delta t \tag{2.1}$$

where δt is a positive infinitesimal, defines a special curvature collineation provided the space F_n admits a vector field $v^i(x)$ with respect to which

$$\mathfrak{L}_v H^i_{jkh}(x , \dot{x}) = 0 . \tag{2.2}$$

Taking Lie derivative of (1.7), using relation (1.10) for the tensor $K^i_{\gamma kh}$ and the fact that $\mathfrak{L}_v \dot{x}^i = 0$ we have

$$\mathfrak{L}_v H^i_{jkh} = \mathfrak{L}_v K^i_{jkh} + \dot{x}^\gamma \dot{\partial}_j (\mathfrak{L}_v K^i_{\gamma kh}) . \tag{2.3}$$

Hence we have

THEOREM (2.1). *In a F_n every curvature collineation is a special curvature collineation.*

The converse of the above theorem is not in general true. In order that this may be true, we have a following sufficient condition.

THEOREM (2.2). *If a F_n admits a special curvature collineation with respect to a vector field v^i and*

$$\mathfrak{L}_v K^i_{jkh} = f(x^i) \tag{2.4}$$

is a function independent of $\dot{x}^i$, then the space admits a curvature collineation.

From (1.11) and (1.14) we may state that

THEOREM (2.3). *In a F_n every affine motion is a special curvature collineation.*

From the above theorem and the lemmas (1.1) and (1.2) we have

THEOREM (2.4). *In a F_n every motion is a special curvature collineation.*

THEOREM (2.5). *In a F_n every homothetic motion is a special curvature collineation.*

We shall now study the condition under which a projective motion is a special curvature collineation. Assuming that the space admits a projective motion and using equations (1.15) and (1.11) we obtain

$$\mathfrak{L}_v H^i_{hjk} = -\delta^i_j P_{h(k)} - \delta^i_h P_{j(k)} - P_{jh(k)} \dot{x}^i + \delta^i_k P_{h(j)} + \delta^i_h P_{k(j)} + P_{kh(j)} \dot{x}^i \tag{2.5}$$

where we used the fact $\dot{x}^i_{(k)} = 0$ and

$$G^i_{jkl} \dot{x}^j = 0 . \tag{2.6}$$

If a projective motion for a space is a special curvature collineation we have from (2.5) and (2.2)

$$-\delta^i_j P_{h(k)} - \delta^i_h P_{j(k)} - P_{jh(k)} \dot{x}^i + \delta^i_k P_{h(j)} + \delta^i_h P_{k(j)} + P_{kh(j)} \dot{x}^i = 0 . \tag{2.7}$$

Contracting the above equation with respect to i and h we get

$$(n+1)(P_{k(j)} - P_{j(k)}) + P_{kh(j)} \dot{x}^h - P_{jh(k)} \dot{x}^h = 0 . \tag{2.8}$$

which after using the relations (1.12) and (2.6) yields

$$(n+1)(P_{k(j)} - P_{j(k)}) + \dot{\partial}_h (P_{k(j)} - P_{j(k)}) \dot{x}^h = 0 . \tag{2.9}$$

Since $P(x, \dot{x})$ is homogeneous of degree one in $\dot{x}^h$, $(P_{k(j)} - P_{j(k)})$ will be homogeneous of degree zero in $\dot{x}^i$ and hence

$$\dot{\partial}_h (P_{k(j)} - P_{j(k)}) \dot{x}^h = 0 . \tag{2.10}$$

Substituting (2.10) in (2.9) we get $P_{k(j)} - P_{j(k)} = 0$. Thus we have

THEOREM (2.6). *A necessary condition for a projective motion to be a special curvature is*

$$P_{k(j)} - P_{j(k)} = 0 . \tag{2.11}$$

The converse of the theorem (2.6) is not always true. However, the projective motion satisfying $P_{k(j)} = 0$ will be a special curvature collineation. For if we substitute $P_{k(j)} = 0$ in (2.5) we get

$$\mathfrak{L}_v H^i_{hjk} = (P_{kh(j)} - P_{jh(k)})\, \dot{x}^i . \tag{2.12}$$

Applying the relation (1.12) for P_k and P_j and using (2.6) we see that right hand side of (2.12) vanishes.

Since the operations of contraction and the Lie derivative are commutative we observe after contracting the indices i and h in (2.2) that every special curvature collineation vector v^i satisfies

$$\mathfrak{L}_v H_{jk} = 0 \tag{2.13}$$

where H_{jk} is given by (1.9).

If a F_n admits a vector field v^i such that (2.13) holds then we shall say that F_n admits a special Ricci collineation. Hence we have

THEOREM (2.7). *In a F_n every special curvature collineation is a special Ricci collineation.*

The converse of this theorem is not true in every Finsler space. However, we shall see in theorem (2.10) below that the converse is true in a special space called isotropic Finsler space.

Contracting (2.3) with respect to i and h we can see that

THEOREM (2.8). *In a F_n every Ricci collineation is a special Ricci collineation.*

Now we shall prove the following theorem

THEOREM (2.9). *A necessary and sufficient condition that a projective motion in a Finsler space be a special Ricci collineation is that*

$$nP_{h(j)} - P_{j(h)} - \dot{\partial}_h \dot{\partial}_j P_{(i)} \dot{x}^i = 0 \tag{2.14}$$

Proof. Let the Finsler space F_n admit a projective motion. Contracting (2.5) with respect to i and k and using the relation (1.12) for P_j and P_k we get

$$\mathfrak{L}_v H_{hj} = nP_{h(j)} - P_{j(h)} - \dot{\partial}_h \dot{\partial}_j P_{(i)} \dot{x}^i = 0 \tag{2.15}$$

This shows that (2.14) is a necessary and sufficient condition that a projective motion be a special Ricci collineation.

THEOREM (2.10). *In an isotropic Finsler space every special Ricci collineation is a special curvature collineation.*

Proof. Since for an isotropic Finsler space $W^i_h = 0$ (lemma (1.3)) we have from (1.8 b)

$$H^i_{jkh} = \frac{1}{n+1}(H_{hk} - H_{kh})\, \delta^i_j + \frac{1}{n+1}(\dot{\partial}_j H_{hk} - \dot{\partial}_j H_{kh})\, \dot{x}^i +$$

$$+ \frac{1}{n^2-1}(nH_{jk} + H_{kj} + \dot{x}^\gamma \dot{\partial}_j H_{k\gamma})\, \delta^i_h - \frac{1}{n^2-1}(nH_{jh} + H_{hj} + \dot{x}^\gamma \dot{\partial}_j H_{h\gamma})\, \delta^i_k .$$

Taking Lie derivative of the above equation, using the relation (1.10) and the fact that $\mathfrak{L}_v \dot{x}^i = 0$ we get

$$\begin{aligned}(2.16)\qquad \mathfrak{L}_v \mathrm{H}^i_{jkh} = {} & \frac{1}{n+1} (\mathfrak{L}_v \mathrm{H}_{hk} - \mathfrak{L}_v \mathrm{H}_{kh})\, \delta^i_j + \\ & + \frac{1}{n+1} (\dot{\partial}_j \mathfrak{L}_v \mathrm{H}_{hk} - \dot{\partial}_j \mathfrak{L}_v \mathrm{H}_{kh})\, \dot{x}^i + \\ & + \frac{1}{n^2-1} (n \mathfrak{L}_v \mathrm{H}_{jk} + \mathfrak{L}_v \mathrm{H}_{kj} + \dot{x}^\gamma \dot{\partial}_j \mathfrak{L}_v \mathrm{H}_{k\gamma})\, \delta^i_h - \\ & - \frac{1}{n^2-1} (n \mathfrak{L}_v \mathrm{H}_{jh} + \mathfrak{L}_v \mathrm{H}_{hj} + \dot{x}^\gamma \dot{\partial}_j \mathfrak{L}_v \mathrm{H}_{h\gamma})\, \delta^i_k .\end{aligned}$$

Now if the space admits a special Ricci collineation (i.e. $\mathfrak{L}_v \mathrm{H}_{hk} = 0$) then we have from (2.16) $\mathfrak{L}_v \mathrm{H}^i_{jkh} = 0$. This proves the theorem.

REFERENCES

[1] G. H. KATZIN, J. LEVINE and W. R. DAVIS, *Curvature collineation: a fundamental symmetry property of the space time of general relativity defined by vanishing Lie derivative of Riemann curvature tensor*, « Journal of Mathematical Physics », *10* (1969), 617–629.

[2] B. N. PRASAD, *Curvature collineations in Finsler space*, « Atti della Accad. Naz. Lincei Rendiconti », *49* (1970), 194–198.

[3] L. P. EISENHART, *Riemannian Geometry*. Princeton University Press (1926).

[4] K. YANO, *The theory of Lie derivatives and its applications*. North Holland Publishing Co., Amsterdam 1957.

[5] H. RUND, *The differential geometry of Finsler spaces*. Springer Verlag 1959.

[6] R. S. SINHA, *On projective motions in a Finsler space with recurrent curvature*, « Tensor N. S. », *21* (1970), 124–126.

[7] H. HIRAMATU, *Groups of homothetic transformations in a Finsler space*, « Tensor N.S. », *3* (1954), 131–143.