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On Volterra's integral equation

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Equazioni integrali. — *On Volterra's integral equation.* Nota di
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Riassunto. — Se in un'equazione integrale di Volterra l'integrazione è considerata nel senso di Perron, e se è soddisfatta una condizione generalizzata di Lipschitz, allora condizione necessaria e sufficiente perchè valga il teorema di esistenza e di unicità è che la successione delle approssimazioni successive di Picard sia equiassolutamente continua.

1. Introduction

In this paper we consider the Volterra integral equation

$$y(x) = (\mathrm{P})\int_0^x \mathrm{F}(x, t, y(t))\,\mathrm{d}t. \tag{1}$$

where $x \in \mathrm{I} = \{x \mid 0 \leq x \leq 1\}$ and $(\mathrm{P})\int$ denotes integration in the Perron sense. We show, using the Perron definition of the integral and a generalized Lipschitz condition, that a Picard sequence yields a unique solution of equation (1) if and only if a sequence is EAC (equi-absolutely continuous) on I. It has been shown (see for example, Bauer [1], McShane [4], or Saks [5]) that the Perron definition of the integral leads to a generalization of the Lebesgue integral. Manougian in [2] used the Perron integral and established an existence and uniqueness theorem for a nonlinear differential equation.

We assume the basic properties of the Perron integral (see [4] or [5]). In particular, we state

Theorem 1. – *If* $g \in \mathrm{P}$ *(Perron integrable) on* I, *then* $\mathrm{G}(x)$ *is continuous and* LAC *(locally absolutely continuous) on* I *where*

$$\mathrm{G}(x) = (\mathrm{P})\int_0^x g \qquad \text{for } x \in \mathrm{I}.$$

Theorem 2. – *If* $g \in \mathrm{P}$ *on* I *and* $g(x) \geq 0$ *a.e. (almost everywhere) on* I, *then* $g \in \mathfrak{L}$ *(Lebesgue integrable) on* I *and*

$$(\mathrm{P})\int_0^x g = (\mathfrak{L})\int_0^x g \qquad \text{for } x \in \mathrm{I}$$

where $(\mathfrak{L})\int$ *denotes integration in the Lebesgue sense.*

(*) Nella seduta del 9 gennaio 1971.

2. MAIN THEOREM

Definition. – Suppose $\{y_n\}$ is a sequence of functions defined on I. Then for each x on I we define

$$F_n(t) = F(x, t, y_n(t)).$$

The following lemma is used in the proof of the main theorem.

LEMMA: *Suppose*

(i) $f_n \in P$ *on* I *for each positive integer* n

(ii) $\lim_{n\to\infty} f_n(x) = f(x)$ *a.e. on* I

(iii) $f_n(x) \geq g(x)$ *a.e. on* I *for each* n, *where* $g \in P$ *on* I.

Then, $f \in P$ *on* I, *and* $\lim_{n\to\infty} (P)\int_0^x f_n = (P)\int_0^x f$ *if and only if the sequence* $\left\{(P)\int_0^x (f_n - g)\right\}$ *is* EAC *on* I.

The proof of this lemma (see Manougian [3]) is based on a corresponding theorem by Vitali [6] for a sequence of functions integrable in the Lebesgue sense.

We now prove the following theorem:

MAIN THEOREM. – *Suppose*

(H1) $F(x, t, y)$ *is a function continuous in* y *for* $x \in I$ *and* $t \in I$.

(H2) $F(x, t, y(t)) \in P$ *in* t *on* I *for* y *continuous on* I.

(H3) $F(x, t, y(t)) \geq g(t)$ *a.e. on* I *for each* x *on* I, *where* $g \in P$ *on* I.

(H4) $|F(x, t, y(t)) - F(x, t, y^*(t))| \leq v(t)\,|y(t) - y^*(t)|$ *a.e. on* I *where* $v \in P$ *on* I.

Then, there exists a Picard sequence yielding a solution, $\varphi(x)$, for equation (1), which is continuous and LAC on I, only if the sequence $\left\{(P)\int_0^x [F_n - g]\right\}$ is EAC on I.

Proof: Define

$$y_0(x) = 0 \tag{2}$$

and

$$y_n(x) = (P)\int_0^x F(x, t, y_{n-1}(t))\,dt \tag{3}$$

for $n = 1, 2, \cdots$. We note that by Theorem 1, $y_1(x)$ is continuous and LAC on I. By induction, $y_n(x)$ is continuous and LAC on I for each positive integer n.

Let

$$u_{n-1}(x) = y_n(x) - y_{n-1}(x) \qquad (n = 1, 2, \cdots) \tag{4}$$

Then,

$$u_0(x) = (\mathrm{P}) \int_0^x \mathrm{F}(x, t, 0)\, \mathrm{d}t$$

and

$$u_n(x) = (\mathrm{P}) \int_0^x [\mathrm{F}_n(t) - \mathrm{F}_{n-1}(t)]\, \mathrm{d}t. \tag{5}$$

By hypothesis (H4) and Theorem 2 the integral in (5) may be taken in the Lebesgue sense. Also, $u_0(x)$ is continuous on I, so that there exists a number k such that for x on I we have

$$|u_0(x)| < k$$

and

$$|u_1(x)| < k\, (\mathfrak{L}) \int_0^x v$$

and in general, since $v(t) \geq 0$ a.e. on I we have

$$|u_n(x)| < \frac{k}{n!} \left[(\mathfrak{L}) \int_0^x v \right]^n$$
$$\leq \frac{k}{n!} \left[(\mathfrak{L}) \int_0^1 v \right]^n .$$

Therefore, $\sum_{i=0}^{\infty} |u_i(x)|$ converges uniformly on I.

But, $\sum_{i=0}^{n} u_i(x) = y_{n+1}(x)$.

Hence, there exists a function $\varphi(x)$ which is continuous and LAC on I such that

$$\lim_{n\to\infty} y_n(x) = \varphi(x) \quad \text{on I.}$$

Now consider,

$$\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} (\mathrm{P}) \int_0^x \mathrm{F}_n(t)\, \mathrm{d}t .$$

By hypothesis (H1) and the Lemma above, we have

$$\varphi(x) = (\mathrm{P}) \int_0^x \mathrm{F}(x, t, \varphi(t))\, \mathrm{d}t$$

only if the sequence $\left\{ (\mathrm{P}) \int_0^x [\mathrm{F}_n(t) - g(t)]\, \mathrm{d}t \right\}$ is EAC on I. Thus the existence of solution of equation (1) is established.

To prove uniqueness, let $\varphi^*(x)$ be another solution of (1) and let

$$Y(x) = \varphi^*(x) - \varphi(x) \qquad \text{for } x \text{ on I}$$

Then,

$$Y(x) = (P)\int_0^x [F(x, t, \varphi^*(t)) - F(x, t, \varphi(t))]\, dt$$

and

$$0 \leq |Y(x)| \leq \lim_{n\to\infty} \frac{k}{n!}\left[(\mathfrak{L})\int_0^1 v\right]^n .$$

Hence, $Y(x) \equiv 0$ on I and uniqueness of the solution of (1) follows.

3. REMARK

Suppose x and t are real numbers on I. Let $\bar{y}(x)$ be a vector function, $\bar{y}(x) = [y_1, y_2, \cdots, y_m]$, in the space of m-dimensions, and $\bar{F}(x, t, \bar{y}(t))$ be a vector function, $\bar{F}(x, t, \bar{y}(t)) = [F_1(x, t, y_1(t)), F_2(x, t, y_2(t)), \cdots, F_m(x, t, y_m(t))]$. Then, the results of the theorem above still hold.

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